MATHEMATICS FOR AGRICULTURE AND ELEMENTARY SCIENCE

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WENTWORTH-SMITH MATHEMATICAL SERIES

MATHEMATICS FOR AGRICULTURE AND ELEMENTARY SCIENCE

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PREFACE

The purpose of this book is to supply the practical mathematics needed in elementary college courses in the sciences, with particular emphasis upon the needs of students of agriculture and forestry. Although written primarily to meet the needs of freshman students in colleges of agriculture and other institutions of collegiate grade offering a brief course in general mathematics as a preliminary to other courses, it should also be found useful in the upper classes of high schools or in junior colleges where it is desired to give students an insight into the utilitarian aspects of the mathematics lying beyond the range covered by the ordinary high-school course.

The book presupposes the amount of algebra and plane geometry usually studied in the high school, but it recognizes the fact that students often forget the technique of these subjects, — particularly algebra. The book therefore begins with a brief review of algebraic operations, linear and quadratic equations, and graphs, with the work strictly limited to those phases that will be needed in the subsequent chapters. The necessary facts of geometry are reviewed from time to time as they are required in the development of the subject.

In the applications of the various mathematical processes the needs of the general student as well as those of the student of agriculture have been carefully considered, so that by the simple expedient of omitting topics not vital under local conditions the book can be adapted to various types of courses. The general nature of the material will be better understood by an examination of the Contents on page v. Special attention is called to the practical nature of the work in functions, logarithms, statistics, investments, variation, and trigonometry, and to

the separation of the more specialized applications in the chapters on dairy mathematics, farm management, and agricultural engineering. As this is essentially a textbook on mathematics, no attempt has been made to cover such fields as mechanical drawing, surveying, physics, and the like. Supported by explanatory notes and illustrative examples, however, it draws freely upon these fields for problems which show the practical applications of mathematics and which stimulate the student's interest.

Particular attention is also called to the list of useful tables at the end of the book. Although inserted primarily as aids in computation in connection with the work of the various chapters, they will be found helpful in general computations and in increasing the value of the book as a source of practical problem material and general information.

In connection with the chapters on dairying, farm management, and agricultural engineering, which the authors believe are unique in mathematical texts of this class, grateful acknowledgment is made of the assistance rendered by several colleagues in the educational field, particularly Professors Francis W. Peck, Robert M. Washburn, and Edward G. Cheyney.

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MATHEMATICS FOR AGRICULTURE AND ELEMENTARY SCIENCE

CHAPTER I

REVIEW OF ALGEBRAIC OPERATIONS

Need for Review. Students often find, upon resuming the study of mathematics in college or in advanced technical classes in high schools, that they have forgotten many of the terms, fundamental laws, operations, and methods of solving problems learned in elementary algebra. This is, of course, to be expected, since we all tend to forget processes which we have not been using, but it shows how necessary it is to have a brief review of the topics that will be needed in applying mathematics to work in agriculture, forestry, or elementary science. In this chapter, therefore, it is proposed to review the operations of elementary algebra that are important in subsequent work.

Term. The word *term* is generally used to indicate the parts of an algebraic expression that are connected by plus or minus signs, but it may be applied to an expression standing alone.

For example, the expression $4a^2 - 4a + 1$ is composed of the three terms $4a^2$, $4a$, and 1, although in practice the second term is usually stated as $-4a$. Similarly, $(a+b)^2 - 2(a+b) + b^2$ has the three terms $(a+b)^2$, $-2(a+b)$, and b^2, the parentheses designating $(a+b)^2$ and $-2(a+b)$ as single terms. Expressions like a or x^2y consist of a single term.

A term may be either numerical or algebraic, and we also see, as in the second of the preceding illustrations, that a term like $2(a+b)$ may contain plus or minus signs. In the work that we shall take up, however, distinctions of this sort are of little importance.

Factors. A *factor* is an expression which is contained in another without a remainder, but 1 is not ordinarily considered as a factor of an expression.

An expression which cannot be factored is said to be *prime.*

For example, 2, 4, 6, 8, and 12 are factors of 24, but only the 2 is prime. Similarly, 3, x, y, x^2, y^2, y^3, xy, x^2y^2, xy^2, and other combinations of these expressions are factors of $3x^2y^3$, and a, $x+3a$, and $x+2a$ are factors of $ax^2+5a^2x+6a^3$. Since $2\frac{1}{2}$ is not a whole number, we do not usually consider it to be a factor of 5. In advanced algebra, however, we sometimes speak of $2\frac{1}{2}$ as a factor of $2\frac{1}{2}x+5$, or of $x-\sqrt{2}$ as a factor of x^2-2.

Monomial and Polynomial. An expression that consists of a single term is called a *monomial,* and one that consists of two or more terms is called a *polynomial.*

Polynomials are classified according to the number of terms, a polynomial of two terms being called a *binomial* and one of three terms a *trinomial.*

For example, a, $\frac{1}{2}xy$, and $\frac{1}{2}x(a+b)$ are all monomials, the last being regarded as a monomial because it expresses the product of factors even though one of the factors is a binomial. Similarly, $\frac{1}{2}a+3$ and $a-\sqrt{b+c}$ are binomials, and $x^2-2xy+y^2$ is a trinomial.

Exercises. Terms, Factors, and Polynomials

1. Name the terms in the trinomial $x^4+2x^2y^2+y^4$.

2. Name four factors of the second term in Ex. 1.

3. Name the terms in the polynomial $a^3-3a^2b+3ab^2-b^3$, using the minus sign before the second and fourth terms.

In each of the polynomials in Exs. 4–6 below, name three terms and also name three factors of each term:

4. $(m+n)mn+2mn+5m^2n+7ab(m+n)$.

5. $(x-2xy+3z)^2k+(x-y)k^2-7(x+y)(x-y)+abcd$.

6. $(a-b)^3-3(a-b)^2(x+y)+3(a-b)(x+y)^2-(x+y)^3$.

7. In the expression $a^2+b\sqrt{a^2-b^2}+c$, is $\sqrt{a^2-b^2}$ a factor of the polynomial? of any term? Is it a monomial? a binomial?

Algebraic Signs. As is learned in elementary algebra, the signs + and − are used in two senses: (1) to indicate the operations of addition and subtraction respectively, and (2) to indicate the quality of a term; that is, whether it is positive or negative. In the latter case it is not in general customary, except for purposes of emphasis, to write the + sign before a term that stands alone, but the − sign is always written before a negative term.

Laws of Signs. The following examples illustrate the laws of signs in the common operations:

ADDITION

$$a + 2a = 3a \qquad a + (-2a) = -a$$
$$-a + 2a = a \qquad -a + (-2a) = -3a$$

SUBTRACTION

$$a - 2a = -a \qquad a - (-2a) = 3a$$
$$-a - 2a = -3a \qquad -a - (-2a) = a$$

MULTIPLICATION

$$3a \cdot 4b = 12ab \qquad (-3a) \cdot 4b = -12ab$$
$$3a \cdot (-4b) = -12ab \qquad (-3a) \cdot (-4b) = 12ab$$

DIVISION

$$\frac{4ab}{2a} = 2b \qquad \frac{-4ab}{2a} = -2b \qquad \frac{4ab}{-2a} = -2b \qquad \frac{-4ab}{-2a} = 2b$$

The laws illustrated above are often stated briefly as follows:

1. *To add two numbers which have like signs, find the sum of their numerical values and prefix to it the common sign.*

2. *To add two numbers which have unlike signs, find the difference between their numerical values and prefix to it the sign of the number which has the greater numerical value.*

3. *To subtract any number, change its sign and add.*

4. *In multiplication and division two like signs produce plus* (+)*; two unlike signs produce minus* (−).

Coefficient. Any factor of an expression is called the *coefficient* of the product of the other factors.

That is, in the expression ab, according to this definition, a is the coefficient of b, and b is the coefficient of a. It is customary, however, to speak of the first factor (a) as the coefficient of the second, and we almost never speak of the second (b) as the coefficient of the first.

Similarly, in the expression $(2\,a + 3\,b)y^2$, $2\,a + 3\,b$ is regarded as the coefficient of y^2, 2 as the coefficient of a, and 3 as the coefficient of b.

We also extend the definition of coefficient to include fractions; that is, in the expression $\frac{fL}{d}$ we may speak of f as the coefficient of $\frac{L}{d}$, or of $\frac{f}{d}$ as the coefficient of L.

Exponent. In the expression a^3, which means $a \cdot a \cdot a$, we speak of 3 as the *exponent* of a. It tells how many times a is taken as a factor.

Similarly, in the case of $3(a + b)^2$, 2 is the exponent of $(a + b)$, but not of 3. If we wish to include the 3, we write $[3(a + b)]^2$.

The terms "coefficient" and "exponent," which are in no sense interchangeable, should not be confused. It should be remembered that

The coefficient indicates the number of addends;

The exponent indicates the number of factors.

That is, $3\,a$ always means $a + a + a$, and a^3 always means $a \cdot a \cdot a$.

Exercises. Coefficients and Exponents

1. In the expression $(3\,x^2 + 5\,b)^3$ name each coefficient and each exponent, and state what each one signifies.

2. In the expression $5\,V^2$, which is the coefficient and which the exponent? What does each one signify?

3. In $2(a + b)(x + y)^3$, of what is 2 the coefficient? Of what is $2(a + b)$ the coefficient? Of what is 3 the exponent?

4. In the expression $V^2 = kH$, what does the 2 indicate? What does the k indicate?

5. What is the difference in meaning of $\frac{V^2}{k}$, $\left(\frac{V}{k}\right)^2$, and $2 \cdot \frac{V}{k}$?

Exercises. Review

1. Add $+40$, -50, $+18$, -22, -58, $+78$, and -82.

2. Write the value of $(-1)+(-1)$; of $(-1)-(-1)$; of $(-1)\times(-1)$; of $(-1)\div(-1)$.

3. Write the value of $(+4)(+5)$; of $(-4)(+5)$; of $(+4)(-5)$; of $(-4)(-5)$.

4. Write the value of $(+8)\div(+2)$; of $(+12)\div(-3)$; of $(-18)\div(+9)$; of $(-20)\div(-10)$; of $(-93)\div(-31)$.

State all the prime factors in each of the following cases:

5. a^3.
6. a^5.
7. $abcd$.
8. $a(b+c)$.
9. $17\,x^2y^3$.
10. $23\,m^4$.
11. $4\,x^2$.
12. $5\,y^3(x+z)$.

In the following cases state one coefficient and each exponent:

13. $28\,x^5$.
14. $31\,y^3$.
15. $14(a+b)^2$.
16. $x^3(a-b)$.
17. $144\,x^5$.
18. $153\,a^3b^4$.
19. xy^3z^5.
20. $-3\,ab^4c^5$.

State rapidly the value of each of the following expressions:

21. $20-6$; $20-(10-4)$; $20-10+4$; $15-(10-5)$.

22. $14-(6-8)$; $14-(6+8)$; $14-(-6-8)$; $15-10+5$.

Copy and complete each of the following statements:

23. $5\,a$ means _____.
24. x^3 means _____.
25. $3\,x^2$ means _____.
26. $2\,a^3$ means _____.

In the following cases write all the monomials (except those in the polynomials) in one column, all the binomials in a second, all the trinomials in a third, and all the polynomials (including the binomials and trinomials) in a fourth:

27. xy^2z^3.
28. $x-1$.
29. $2\,x-6$.
30. $4\,x^5+7$.
31. $3\,x^2y^2z^5$.
32. x^3+y+1.
33. $5\,x^3y-2$.
34. x^5-y+z.
35. a^3+b.
36. $2+a+b+x$.
37. $a-b+c-d^2$.
38. $\frac{1}{3}\,x+y+z+w$.

Square of a Binomial. In algebra there are several special products that are of great importance. The first of these products is the square of a binomial, a type that may be represented by

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

The sign $\pm$ shows that the two cases, $(a + b)^2$ and $(a - b)^2$, are involved. The $+$ parts of the two $\pm$ signs go together, and so do the $-$ parts; that is,

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \text{and} \quad (a - b)^2 = a^2 - 2ab + b^2.$$

Thus, $(3x + y)^2 = (3x)^2 + 2(3x)y + y^2 = 9x^2 + 6xy + y^2,$

and $(x + y - z)^2 = [(x + y) - z]^2 = (x + y)^2 - 2(x + y)z + z^2.$

This last result may be expanded into $x^2 + 2xy + y^2 - 2xz - 2yz + z^2$, but this is not usually done unless there is some special reason for it.

Product of the Sum and Difference. The next product is that of the sum and difference of two expressions; namely,

$$(a + b)(a - b) = a^2 - b^2.$$

Thus, $(2x + 3y)(2x - 3y) = (2x)^2 - (3y)^2 = 4x^2 - 9y^2,$

and $(x - y + z)(x + y - z) = [x - (y - z)][x + (y - z)] = x^2 - (y - z)^2.$

As above, this last result may be expanded if necessary.

Binomials with a Common Term. The third product is that of two binomials with a common term, the type form of which is

$$(a + b)(a + c) = a^2 + (b + c)a + bc.$$

Thus, $(x + 3)(x + 7) = x^2 + (3 + 7)x + 3 \cdot 7 = x^2 + 10x + 21,$

$(x + 3)(x - 7) = x^2 + (3 - 7)x + 3(-7) = x^2 - 4x - 21,$

and $(x - 3)(x - 7) = x^2 + [(-3) + (-7)]x + (-3)(-7) = x^2 - 10x + 21.$

Binomials with a Common Factor. When the binomials have a common factor in one term, the product is of the form

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$$

Thus, $(2x + 3)(5x - 7) = 2 \cdot 5x^2 + [2(-7) + 3 \cdot 5]x + 3(-7)$

$$= 10x^2 + x - 21.$$

A simple check upon such operations is to substitute convenient values for the letters. Letting $x = 1$ in the last case above, we have

$$(2x + 3)(5x - 7) = (2 + 3)(5 - 7) = 5 \cdot (-2) = -10,$$

and $10x^2 + x - 21 = 10 + 1 - 21 = -10.$

Exercises. Special Products

In each of the following cases write the result without actually multiplying:

1. $(x+3)^2$.
2. $(x-3)^2$.
3. $(3-x)^2$.
4. $(x-a)^2$.
5. $(a-x)^2$.
6. $(3x+1)^2$.
7. $(3x-1)^2$.
8. $(1-3x)^2$.
9. $(3x+7)^2$.
10. $(3x \pm 5)^2$.
11. $(2x+3y)^2$.
12. $(2x-3y)^2$.
13. $(3y-2x)^2$.
14. $(3x+2y)^2$.
15. $(3x-\frac{1}{3}y)^2$.

Write the results of the following operations and check each result twice, first by letting $x = 1$ and then by letting $x = 2$:

16. $(x+5)^2$.
17. $(2x+1)^2$.
18. $(2x+5)^2$.
19. $(5-2x)^2$.
20. $(x+\frac{1}{4})^2$.
21. $(2x-\frac{1}{3})^2$.

In each of the following cases write the result and check it:

22. $[(x-3)+a]^2$.
23. $[x-(3+a)]^2$.
24. $[x+(3-a)]^2$.
25. $(m+1)(m-1)$.
26. $(1+m)(1-m)$.
27. $(3+k)(3-k)$.
28. $(4x+3)(4x-3)$.
29. $(8k+m)(-m+8k)$.
30. $(5v-3)(5v+3)$.
31. $(5v-3)(5v+2)$.
32. $(2x+3)(4x+7)$.
33. $(4x+a)(3x-b)$.
34. $(3x-4)(2x+5)$.
35. $(ax-d)(bx-c)$.
36. $(7x-3)(2x-3)$.
37. $[(a+b)+c][(a+b)-c]$.
38. $[a+(b+c)][a-(b+c)]$.
39. $(a+b+c)(a-b+c)$.
40. $(a+b-c)(a-b+c)$.
41. $(V-K+a)(V-K+a)$.
42. $(PV-K+7c)(PV-K-4c)$.
43. $(x^2-x+1)(x^2+x-1)$.
44. $(2x^2-x+1)(2x^2-x-1)$.
45. $[(a+b)x-1][(a+b)x+1]$.
46. $(a+bx-1)(a+bx+1)$.
47. $[a+b(x-1)][a-b(x-1)]$.
48. $[-\frac{1}{2}(p-\sqrt{p^2-4q})]^2$.
49. $[-\frac{1}{2}(p+\sqrt{p^2-4q})]^2$.
50. $(a-\sqrt{x-1})(a+\sqrt{x-1})$.
51. $(a-\sqrt{x}-1)(a+\sqrt{x}-1)$.

Factoring. The cases of factoring polynomials that the student will need to use in subsequent work may be classified in three general types, as follows:

1. A polynomial with a common factor in each term.

For example, $a + 3\,ab - 3\,ab^2 = a(1 + 3\,b - 3\,b^2)$. Such cases involve simply the factoring of monomials (the terms of the polynomials), and are so easy as to require no special attention.

2. The difference between two squares.

For example, $a^2 - b^2 = (a + b)(a - b)$. This case follows readily as the inverse of the second special product given on page 6; that is, whenever we have to factor such an expression, we simply reverse the special-product formula $(a + b)(a - b) = a^2 - b^2$.

Similarly, in factoring a more complicated form of this case, we have

$$\begin{aligned} a^2 - b^2 + 2\,bc - c^2 &= a^2 - (b^2 - 2\,bc + c^2) \\ &= a^2 - (b - c)^2 \\ &= (a + b - c)(a - b + c). \end{aligned}$$

3. Trinomials of the second degree that are readily factorable by trial.

In the case of the square of a binomial, for example, $x^2 - 4\,x + 4$, we have the inverse of the first formula on page 6; that is, we have

$$x^2 - 4\,x + 4 = x^2 - 2(2)x + 2^2 = (x - 2)^2.$$

We also have the case of the product of two binomials which have a common term. Since $(a + b)(a + c) = a^2 + (b + c)a + bc$, as shown on page 6, we reverse the process in a case like $x^2 + 5\,x + 6$. That is, we find two numbers whose sum, $b + c$, is 5, and whose product, bc, is 6. These numbers are evidently 2 and 3, and hence

$$x^2 + 5\,x + 6 = (x + 2)(x + 3).$$

Finally, we have the case of the product of two binomials with a common factor in one term. Since $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$, we see, by a little experimenting with possible factors, that

$$3\,x^2 - 2\,x - 8 = 3 \cdot 1\,x^2 + [3 \cdot (-2) + 4 \cdot 1]x + 4 \cdot (-2) = (3\,x + 4)(x - 2).$$

The student can easily see that several of the possible pairs of factors are as follows:

$$(3\,x - 4)(x + 2), \quad (3\,x + 2)(x - 4), \quad (3\,x + 4)(x - 2), \quad \text{and} \quad (3\,x - 2)(x + 4).$$

It is, therefore, simply a matter of picking out the pair of factors in which $ad + bc = -2$, which, in this case, is the third pair.

Exercises. Factoring

Factor each of the following expressions:

1. $x^2 - m^2$.
2. $x^2 - m^2n^2$.
3. $9\,x^2 - m^2n^4$.
4. $9\,x^2 - 25\,y^2$.
5. $81\,x^2y^2 - 49$.
6. $a^2 - 2\,a + 1$.
7. $m^2 + 2\,mn + n^2$.
8. $x^2 - 4\,xy^2 + 4\,y^4$.
9. $9\,x^2 + 126\,x + 441$.
10. $16\,p^2 - 24\,pq + 9\,q^2$.

Factor the following expressions and check each result by letting one letter equal 1 and the other 2:

11. $V^2 - aV - 110\,a^2$.
12. $x^2 - xy - 56\,y^2$.
13. $a^2 - 2\,ab - 63\,b^2$.
14. $m^2 + 7\,mn - 18\,n^2$.
15. $p^2 + 12\,pq + 27\,q^2$.
16. $2\,N^2 + 3\,Nw + w^2$.

17. Write the complete factors of the expression $a^4 - b^4$; of the expression $16\,a^4 - 81\,b^4$.

In the first case we have $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b)$.

As in Ex. 17, factor each of the following expressions:

18. $16\,a^4 - 1$.
19. $a^4 - 16$.
20. $a^4 - 81$.
21. $16\,a^4 - 81$.
22. $a^8 - b^8$.
23. $256 - x^8$.

Factor the following expressions and check each result by multiplying the factors together:

24. $7\,k^2 + 9\,k - 10$.
25. $(K^2 + 9)^2 - 36\,K^2$.
26. $4\,b^2 + 8\,b + 3$.
27. $6\,x^2 + 13\,x + 6$.
28. $6\,x^2 - 5\,x - 6$.
29. $7\,gk^2 - 28\,g^2k + 42\,g^3k$.
30. $256\,a^4 + 544\,a^2 + 289$.
31. $15\,P - 5\,PV + V - 3$.
32. $x^2 + y^2 - z^2 + 2\,xy$.
33. $25\,a^2 + 16\,a^4 - 40\,a^3$.

Using fractional or surd factors when necessary, factor each of the following expressions:

34. $a + b - 2\sqrt{ab}$.
35. $\dfrac{x^2}{y^2} + \dfrac{y^2}{x^2} + 2$.
36. $x^2 - 4\,x\sqrt{y} + 4\,y$.
37. $\dfrac{x^2}{y} + \dfrac{2\,x}{\sqrt{y}} + 1$.

Law of Exponents in Multiplication. The law of exponents in multiplication is generally expressed as

$$a^m \cdot a^n = a^{m+n}.$$

In a case like $a^2 \cdot a^3$ it is evident from the definition of exponent that

$$a^2 \cdot a^3 = a \cdot a \times a \cdot a \cdot a = a^5, \quad \text{or} \quad a^{2+3}.$$

The same reasoning holds for any positive integral exponents, as m and n, and the law is also true for zero, negative, and fractional exponents.

Law of Exponents in Division. In cases involving division the law relating to exponents is

$$\frac{a^m}{a^n} = a^{m-n}.$$

As in the case of multiplication, this law is easily shown to be true when both m and n are positive integers and $m > n$. The law is, however, true for all exponents.

Law of Exponents in Powers. In the case of powers the law is

$$a^n b^n = (ab)^n.$$

For example, we have $a^2b^2 = (ab)^2$. That is, from the definition of exponent, we have $(ab)^2 = ab \cdot ab = aa \cdot bb$, and hence $(ab)^2 = a^2b^2$.

Meaning of the Zero Exponent. Mathematicians agree that an expression with a zero exponent shall be equal to 1; that is,

$$a^0 = 1.$$

It is manifestly desirable that the law $\frac{a^m}{a^n} = a^{m-n}$ shall be true for all exponents. Thus, for the case in which $n = m$, we have a^{m-m}, or a^0, for the result. Since $\frac{a^m}{a^m} = 1$, we see why a^0 is defined as equal to 1.

Meaning of a Negative Exponent. A negative exponent, such as $-n$ in a^{-n}, is defined in such a way that

$$a^{-n} = \frac{1}{a^n}.$$

That is, if $\frac{a^m}{a^n} = a^{m-n}$ is to cover all cases, it must include the case of $\frac{a^0}{a^n} = a^{0-n} = a^{-n}$. Since $a^0 = 1$, we have $\frac{a^0}{a^n} = \frac{1}{a^n}$, and hence a^{-n} is defined as the reciprocal of a^n.

Meaning of a Fractional Exponent. It is also obviously desirable that fractional exponents should be defined in harmony with the laws of exponents. If the law $a^m \cdot a^n = a^{m+n}$ is to apply, we must therefore define $a^{\frac{1}{2}}$ in such a way that

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$$

Since $\sqrt{a} \cdot \sqrt{a} = a$, we agree that $a^{\frac{1}{2}} = \sqrt{a}$, or in general that

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

In other words, in using fractional exponents we agree that

The numerator indicates a power and the denominator a root.

Laws of Exponents in Roots. In a review of this kind it is not necessary to go into the reasons for the various laws and definitions. We shall, therefore, simply summarize the laws of exponents relating to roots, expressing them with both radical signs and fractional exponents:

With Radical Signs	With Fractional Exponents
$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$	$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$
$\sqrt[n]{a^m} = (\sqrt[n]{a})^m$	$(a)^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$
$\sqrt[n]{a^{mn}} = a^m$	$(a^{mn})^{\frac{1}{n}} = a^m$
$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$

The following examples illustrate the applications of the above laws:

$\sqrt[3]{27\,x^3} = \sqrt[3]{27} \cdot \sqrt[3]{x^3} = 3\,x$	$(27\,x^3)^{\frac{1}{3}} = 27^{\frac{1}{3}}(x^3)^{\frac{1}{3}} = 3\,x$
$\sqrt[4]{x^8} = (\sqrt[4]{x})^8 = x^2$	$x^{\frac{8}{4}} = \left(x^{\frac{1}{4}}\right)^8 = x^2$
$\sqrt[3]{8\,x^2y^7} = \sqrt[3]{8\,y^6 \cdot x^2y} = 2\,y^2\sqrt[3]{x^2y}$	$(8\,x^2y^7)^{\frac{1}{3}} = (8\,y^6 \cdot x^2y)^{\frac{1}{3}} = 2\,y^2\left(x^{\frac{2}{3}}y^{\frac{1}{3}}\right)$
$\sqrt{\frac{4\,x^2}{y^2}} = \frac{\sqrt{4\,x^2}}{\sqrt{y^2}} = \frac{2\,x}{y}$	$\left(\frac{4\,x^2}{y^2}\right)^{\frac{1}{2}} = \frac{(4\,x^2)^{\frac{1}{2}}}{(y^2)^{\frac{1}{2}}} = \frac{2\,x}{y}$

The chief application of reductions of this type will be found in the formulas that are used in making measurements in geometry and physics.

In simple cases it is generally easier to use the radical sign, but in complicated expressions the fractional exponents are frequently clearer.

Exercises. Laws of Exponents

Perform each of the following indicated operations:

1. $a^2 \cdot a^8$.
2. $a^2 \cdot a^{-8}$.
3. $\dfrac{a^2b^8}{ab^2}$.
4. $a^0 \cdot a^5$.
5. $a^{-5} \cdot a^5$.
6. $\dfrac{a^5b}{ab^{-5}}$.
7. $(a^2b^2)^3$.
8. $(-a^2)^3$.
9. $\dfrac{-a^3}{-a^2}$.
10. $(-a^{-a}) \cdot (-a^a)$.
11. $(a^{-a}) \cdot (a^{-a})$.
12. $\dfrac{a^0b^0c^0}{a^{-1}b^{-1}c^{-1}}$.

13. Find the value of a^n when $a = 9$, $m = 3$, and $n = 2$. Which is the easier, to find the mth power of the nth root or the nth root of the mth power?

14. Express $a^{-\frac{m}{n}}$ without using either fractional or negative exponents. Find its value when $a = 81$, $m = -1$, and $n = 2$.

15. Express $\dfrac{\sqrt[n]{a^m}}{b}$ in an integral form, using a negative exponent and a fractional exponent instead of the radical sign. Find its value when $a = 5$, $b = 25$, $m = 6$, and $n = 3$.

16. Using the shortest method, find the value of $\sqrt{2} \cdot \sqrt{162}$.

Reduce the following expressions to forms in which there are no negative exponents or common radical signs, perform the indicated operations, and simplify the results:

17. $\dfrac{4^{-2} + 2^{-2}}{2^{-2} - 4^{-2}}$.
18. $\dfrac{8^{\frac{2}{3}} + 4}{2^5}$.
19. $\dfrac{\sqrt[3]{4} \cdot \sqrt{4^5} \cdot 4^{-\frac{2}{3}}}{4^{\frac{3}{5}} \cdot 4^{\frac{3}{2}} \cdot 4^{\frac{1}{15}}}$.
20. $\sqrt{\dfrac{(a-b)^{-3}(a+b)^2}{(a+b)^{-2}(a-b)^{-1}}}$.
21. $\dfrac{(a+b)^{\frac{1}{2}} \cdot (a+b)^{\frac{2}{3}}}{(a+b)^{-\frac{5}{6}}}$.
22. $\dfrac{49\,a^{\frac{7}{2}}b^{\frac{3}{2}}c^4}{2401 \cdot 7^{-2} \cdot a^{-\frac{3}{2}}b \cdot b^{\frac{1}{2}} \cdot c^4}$.

Perform each of the following indicated operations:

23. $(25\,x^4a^{-2}y^6)^{\frac{1}{2}}$.
24. $\dfrac{(3^2x^3y^{-3})^3}{(16^{\frac{1}{2}}x^{\frac{3}{2}}y^{-\frac{1}{3}})^6}$.
25. $\sqrt[n]{x^{mn}y^nz^{-n}}$.
26. $\dfrac{5(a^{\frac{3}{2}}b^{\frac{2}{3}})^2}{(a^{\frac{2}{3}}b^{\frac{3}{2}})^{-2}}$.

Algebraic Fraction. An expression of the form $\frac{a}{b}$, where b does not have a numerical value, is called an *algebraic fraction.*

For example, $\frac{2}{x}$, $\frac{x}{5y}$, $\frac{x+y}{z}$, and $\frac{a^2+b^2}{a^2-b^2}$ are algebraic fractions, while $\frac{x}{2}$, which may be written $\frac{1}{2}x$, is considered to have a numerical coefficient, and is therefore not an algebraic fraction.

Just as $\frac{3}{5}$ means that 3 is to be divided by 5, or that 3 of the 5 equal parts of 1 are taken, so $\frac{a}{b}$ means that a is to be divided by b, or that a of the b equal parts of 1 are taken.

The familiar designations used with arithmetic fractions are applied in algebra with the same meanings.

For example, the fraction $\frac{a}{b}$ has the *numerator* a and the *denominator* b. We also use the terms *proper fraction* and *improper fraction* in much the same way as they are used in arithmetic, but we do not often need them. In algebra we speak of a *mixed expression* instead of a mixed number.

Reduction of Fractions. Algebraic fractions are reduced to lower or higher terms just as in arithmetic; that is,

Multiplying or dividing both numerator and denominator by the same expression does not change the value of the fraction.

For example, just as $\frac{5}{6} = \frac{10}{12}$, so $\frac{x}{y} = \frac{x \cdot z}{y \cdot z} = \frac{xz}{yz}$, and $\frac{x}{z} = \frac{x \cdot xz}{z \cdot xz} = \frac{x^2z}{xz^2}$.

Similarly, just as $\frac{6}{8} = \frac{3}{4}$, so $\frac{a-b}{a^2-b^2} = \frac{\cancel{a-b}}{\cancel{(a-b)}(a+b)} = \frac{1}{a+b}$.

As in arithmetic, a fraction is said to be *reduced to lowest terms* when the numerator and denominator have no common factors; that is,

To reduce an algebraic fraction to lowest terms, divide both numerator and denominator by all their common factors.

For example,
$$\frac{3\,a}{24\,a^2} = \frac{1}{8\,a},$$

$$\frac{x^2y^3}{x^3y^2} = \frac{y}{x},$$

and
$$\frac{x^2-y^2}{x^2-2\,xy+y^2} = \frac{(x+y)\cancel{(x-y)}}{\cancel{(x-y)}(x-y)} = \frac{x+y}{x-y}.$$

Signs of a Fraction. In an algebraic fraction there are three signs to consider, as follows: (1) the sign of the fraction as a whole, (2) the sign of the numerator, and (3) the sign of the denominator.

From the Law of Signs in division (page 3), we have

$$\frac{a}{b} = \frac{-a}{-b} = -\frac{-a}{b} = -\frac{a}{-b}.$$

By applying this law to fractions, we see that

The value of a fraction is not altered (1) *by changing the signs of both numerator and denominator,* (2) *by changing the signs of both fraction and numerator, or* (3) *by changing the signs of both fraction and denominator.*

If there is no sign before a fraction, the plus sign is, of course, understood; that is, $\frac{a}{b}$ means $+\frac{a}{b}$ or $+\frac{+a}{+b}$.

A common source of error is in failing to remember that changing the sign of the numerator of a fraction changes the sign of every term in the numerator, and similarly for the denominator. For example, we should write

$$\frac{2a+b+c}{3a-2b-c} = -\frac{-2a-b-c}{3a-2b-c} = \frac{-2a-b-c}{-3a+2b+c}.$$

Exercises. Signs of a Fraction

Copy the numbers of these examples and after each one write the word "True" or "False" according to which is appropriate:

1. $+\dfrac{a-b}{a+b} = -\dfrac{a+b}{a+b}.$

2. $-\dfrac{2-x}{x+y} = +\dfrac{-2+x}{x+y}.$

3. $+\dfrac{a^2+1}{a^2-1} = -\dfrac{a^2+1}{1-a^2}.$

4. $-\dfrac{x+y}{x-y} = +\dfrac{x+y}{x-y}.$

5. $+\dfrac{-x-y}{x^2+y^2} = -\dfrac{x+y}{x^2+y^2}.$

6. $\dfrac{a-b}{x-y} = \dfrac{b-a}{y-x}.$

7. $\dfrac{a^2+b^2}{a-b} = -\dfrac{a^2-b^2}{a-b} = -(a+b).$

8. $\dfrac{x^2-1}{x-1} = \dfrac{1-x^2}{1+x} = 1-x.$

9. $\dfrac{p^2-4}{p-2} = \dfrac{p^2+4}{p+2} = p+2.$

10. $\dfrac{p^2-4}{-p-2} = \dfrac{p^2-4}{-(p+2)} = -(p-2).$

Lowest Common Denominator. In arithmetic we use the term *least common denominator* to mean the smallest number that contains the denominator of two or more fractions as factors.

For example, the least common denominator of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ is 12, which is the product of every factor that appears in any denominator, each factor being taken the greatest number of times that it occurs in any one. In reducing these fractions to the least common denominator, we therefore have

$$\tfrac{1}{2} = \tfrac{6}{12}, \quad \tfrac{1}{3} = \tfrac{4}{12}, \quad \text{and} \quad \tfrac{1}{4} = \tfrac{3}{12}.$$

In algebra we use the term *lowest common denominator* instead of least common denominator because we are dealing with the degrees of the denominators rather than their values.

For example, the lowest common denominator (usually abbreviated to L.C.D.) of $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$ is xyz, which, like the 12 above, is the product of every factor that appears in any denominator, each factor being taken the greatest number of times that it occurs in any one. Reducing to the L.C.D., we have

$$\frac{1}{x} = \frac{yz}{xyz}, \quad \frac{1}{y} = \frac{xz}{xyz}, \quad \text{and} \quad \frac{1}{z} = \frac{xy}{xyz}.$$

If we let $x = 2$, $y = 3$, and $z = 4$, we have $xyz = 24$, whereas the least common denominator of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ is 12. This shows that the word "lowest" is used in algebra because such a denominator does not necessarily have the least value for all values of the letters.

Illustrative Example. Reduce the fractions $\dfrac{x-y}{x^2+3xy+2y^2}$ and $\dfrac{x}{2x^2+3xy+y^2}$ to fractions having the L.C.D.

Factoring the denominators of the two given fractions, we have

$$x^2 + 3xy + 2y^2 = (x+y)(x+2y)$$

and

$$2x^2 + 3xy + y^2 = (x+y)(2x+y).$$

The L.C.D. of the given fractions is therefore $(x+y)(x+2y)(2x+y)$. Since the denominator of the first fraction contains all these factors except $2x+y$, we multiply both numerator and denominator by $2x+y$. Similarly, in the second fraction we multiply by $x+2y$. We thus have

$$\frac{x-y}{x^2+3xy+2y^2} = \frac{(x-y)(2x+y)}{(x+y)(x+2y)(2x+y)}$$

and

$$\frac{x}{2x^2+3xy+y^2} = \frac{x(x+2y)}{(x+y)(x+2y)(2x+y)}.$$

Such results are usually left in factored form, as shown above.

Exercises. Reduction of Fractions

Reduce the following fractions to lowest terms:

1. $\frac{a^2-b^2}{ab+b^2}$.
2. $\frac{4x^2-9y^2}{2xy+3y^2}$.
3. $\frac{x^2-2x-8}{x^2+3x+2}$.

Reduce the following expressions to improper fractions:

4. $3a+\frac{7a}{b}$.
6. $m^2-\frac{a}{m^2}$.
8. $x-1+\frac{x}{x-1}$.
5. $2x-\frac{3x}{y}$.
7. $n^2+\frac{m^2}{a}$.
9. $x+1-\frac{x}{x-1}$.

Reduce the following fractions to integral or mixed expressions:

10. $\frac{16-9x^2}{4+3x}$.
11. $\frac{16+9x^2}{4-3x}$.
12. $\frac{x^3+y^3}{x^2-xy+y^2}$.

In each of the following cases reduce the fractions to fractions having the L.C.D.:

13. $\frac{x+1}{x-1}, \frac{x-1}{x+1}, \frac{x^2+1}{x^2-1}$.
14. $\frac{a+2b}{a^2+4b^2}, \frac{a-2b}{a+2b}, \frac{a+b}{a^4-16b^4}$.
15. $\frac{x^3+y^3}{x+y}, \frac{x^3-y^3}{x-y}, \frac{x^3+y^3}{x^2-xy+y^2}, \frac{x+y}{x^2+y^2}$.
16. $\frac{x^2-y^2}{x^2+y^2}, \frac{x+y}{x-y}, \frac{x^2+y^2}{x^2-y^2}, \frac{x-y}{x+y}$.
17. $\frac{x+y}{2x+3y}, \frac{x-y}{x}, \frac{x}{3x-y}, \frac{x^2-y^2}{6x^2+7xy-3y^2}$.
18. $\frac{a+b}{a^2+b}, \frac{b+a}{a^2-b}, \frac{a-b}{b+a^2}, \frac{b-a}{b-a^2}$.
19. $\frac{x^2+y^2}{x^4+x^2y^2+y^4}, \frac{x+y}{x^2+y^2+xy}, \frac{x-y}{x^2+y^2-xy}$.
20. $\frac{a^2-2ab+b^2}{a^2+2ab+b^2}, \frac{a^2+2ab+b^2}{a^2-2ab+b^2}, \frac{(a^2+b^2)^2}{(a^2-b^2)^2}$.

Fractions should be reduced to lowest terms before finding the L.C.D.

Operations with Fractions. In algebra the methods of performing operations with fractions are similar to those used in arithmetic, as will be seen from the following examples:

1. Add $\frac{a}{a+b}$, $\frac{-b}{a-b}$, and $\frac{c^2}{a^2-b^2}$.

Reducing to fractions having the L.C.D. a^2-b^2, we have

$$\frac{a}{a+b}+\frac{-b}{a-b}+\frac{c^2}{a^2-b^2}=\frac{a^2-ab}{a^2-b^2}-\frac{ab+b^2}{a^2-b^2}+\frac{c^2}{a^2-b^2}=\frac{a^2-2\,ab-b^2+c^2}{a^2-b^2}.$$

Care should be taken in cases like $-\frac{ab+b^2}{a^2-b^2}$ to change every sign in the numerator; that is, to change the signs of both ab and b^2.

2. Multiply $\frac{3\,a-6\,b}{4\,a+2\,b}$ by $\frac{8(2\,a+b)^2}{4\,a^2-20\,ab+24\,b^2}$.

First factoring the terms of the given fractions, we have

$$\frac{3\,a-6\,b}{4\,a+2\,b}\cdot\frac{8(2\,a+b)^2}{4\,a^2-20\,ab+24\,b^2}=\frac{3\cancel{(a-2b)}}{\cancel{2}\cancel{(2a+b)}}\cdot\frac{\overset{2}{\cancel{8}}\cancel{(2a+b)}(2\,a+b)}{\cancel{4}\cancel{(a-2b)}(a-3\,b)}=\frac{3(2\,a+b)}{a-3\,b}.$$

This solution shows clearly the great advantage of factoring.

3. Divide $\frac{24\,xy-3\,xz}{27\,yz-3\,xy}$ by $\frac{8\,x^2y-x^2z}{9\,y^2z-xy^2}$.

First factoring, so as to see the common factors, and then inverting the divisor and multiplying, as in arithmetic, we have

$$\frac{3\,x(8\,y-z)}{3\,y(9\,z-x)}\div\frac{x^2(8\,y-z)}{y^2(9\,z-x)}=\frac{\cancel{3}\,x\cancel{(8\,y-z)}}{\cancel{3}\,\cancel{y(9\,z-x)}}\cdot\frac{\overset{y}{\cancel{y^2}}\cancel{(9\,z-x)}}{\underset{x}{\cancel{x^2}}\cancel{(8\,y-z)}}=\frac{y}{x}.$$

4. Divide $\frac{1}{a}+\frac{1}{b}$ by $\frac{a^2-b^2}{a^2b^2}$.

After adding the two parts of the dividend, we proceed as follows:

$$\frac{b+a}{ab}\div\frac{a^2-b^2}{a^2b^2}=\frac{\cancel{b+a}}{\cancel{ab}}\cdot\frac{\overset{ab}{\cancel{a^2b^2}}}{\cancel{(a+b)}(a-b)}=\frac{ab}{a-b}.$$

5. Find the square root of $\frac{4\,a^2-4\,ab+b^2}{(2\,a+b)^3}$.

$$\sqrt{\frac{4\,a^2-4\,ab+b^2}{(2\,a+b)^3}}=\sqrt{\frac{(2\,a-b)^2}{(2\,a+b)^2(2\,a+b)}}=\frac{2\,a-b}{2\,a+b}\sqrt{\frac{1}{2\,a+b}}.$$

Exercises. Operations with Fractions

Simplify each of the following expressions:

1. $\dfrac{x+y}{2}+\dfrac{x-y}{2}$.

2. $\dfrac{x+y}{2}-\dfrac{x-y}{2}$.

3. $\dfrac{x-y}{2}-\dfrac{x+y}{2}$.

4. $\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}$.

5. $\dfrac{a-b}{a+b}-\dfrac{a^2-b^2}{a^2+b^2}$.

6. $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}$.

7. $\dfrac{a}{bc}\cdot\dfrac{b}{ca}\cdot\dfrac{c}{ab}\cdot abc$.

8. $\dfrac{x^2-3xy-4y^2}{x+y}$.

9. $\dfrac{(a-b)^2}{a+b}\cdot\dfrac{a^2-b^2}{a^2+b^2}$.

10. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{1}{a}-\dfrac{1}{b}}$.

11. $\dfrac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$.

12. $\dfrac{\dfrac{x^2}{y^2}-\dfrac{y^2}{x^2}}{x+\dfrac{y^2}{x}}$.

13. $\dfrac{a}{bc}-\dfrac{b}{ca}-\dfrac{c}{ab}$.

14. $\dfrac{a^4+a^2b^2+b^4}{a^2+ab+b^2}$.

15. $\dfrac{3a}{3a-2}\cdot\dfrac{9a^2-4}{9a^2}$.

16. $\dfrac{3-\frac{1}{3}}{2\frac{1}{2}}-\dfrac{6-\frac{2}{3}}{5}$.

17. $\dfrac{6ab-(a+2b)^2}{a^3+8b^3}$.

18. $\dfrac{a}{a-2}+\dfrac{5}{a-8}$.

19. $\dfrac{3a}{2b}+\dfrac{3b}{2c}+\dfrac{3c}{2a}$.

20. $\dfrac{a^2+2ab+b^2-c^2}{a+b-c}$.

21. $\dfrac{x^2-2x+1-n^4}{x-n^2-1}$.

22. $\left(a+\dfrac{\dfrac{b}{a}}{a+\dfrac{b}{a}}\right)$.

23. $\dfrac{1+\dfrac{1}{x^2}+\dfrac{1}{x^4}}{1+\dfrac{1}{x}+\dfrac{1}{x^2}}$.

24. $1+\dfrac{\sqrt{R+\dfrac{1}{R}}\cdot\sqrt[3]{\left(R-\dfrac{1}{R}\right)^2}}{\sqrt[3]{\left(R+\dfrac{1}{R}\right)^2}\cdot\sqrt{R-\dfrac{1}{R}}}$.

Exercises. Test on Chapter I

Perform each of the following subtractions:

1. $a+1$ minus $a-1$

2. $x+y$ minus $2y$

3. $8y$ minus $-6+3y$

4. $9(x+y)$ minus $-7(x+y)$

5. $18a$ minus $-13a+8b$

6. $9a-13b-143$ minus $-6a+8b-92$

Perform each of the following divisions:

7. $\frac{x^3}{x}$.

8. $\frac{3y^2}{y}$.

9. $\frac{5a^3}{a^2}$.

10. $\frac{6a+3b}{3}$.

11. $\frac{8x-4y}{2}$.

12. $\frac{20m-30n}{10}$.

13. $\frac{8a^2-6a}{2a}$.

14. $\frac{a^2-b^2}{a+b}$.

15. $\frac{12x^2-4x^3}{-4x^2}$.

Perform each of the following multiplications:

16. $8(3x+5)$.

17. $x(a+b)$.

18. $-5(8x+3)$.

19. $(a+2)(a+5)$.

20. $(2a-5)(2a-5)$.

21. $(3x+2y)(3y+2x)$.

22. $4(3x-6)$.

23. $-4(-2x-3)$.

24. $(x+y)(x+y)$.

25. $(x-y)(x+y)$.

26. $(a+b)(c+d)$.

27. $(a+2b)(2b-a)$.

28. Give two cases showing how the use of negative numbers makes subtraction in algebra possible in places where it would be impossible in arithmetic.

29. Give one example illustrating the use of + and − as signs of operation, and another showing their use as signs of quality.

30. How can it be determined which is algebraically the smaller of two positive numbers? of two negative numbers?

31. Zero is the least of all arithmetic numbers. Is it the least of all algebraic numbers? Give your reasons.

Exercises. General Review

1. Using the shortest method you know, multiply 284 by 10; by 5; by 11; by 9.

Perform the following arithmetic multiplications, using the shortest method you know:

2. 2×5555.

3. 9×4040.

4. 25×364.

5. 125×888.

6. 250×484.

7. $33\frac{1}{3} \times 369$.

Perform the following arithmetic divisions, using the shortest method you know:

8. $\frac{6735}{5}$.

9. $\frac{4070}{110}$.

10. $\frac{6975}{25}$.

11. $\frac{8888}{10}$.

12. $\frac{1{,}111{,}000}{125}$.

13. $\frac{5{,}555{,}000}{625}$.

14. $\frac{22{,}200}{33\frac{1}{3}}$.

15. $\frac{44{,}400}{66\frac{2}{3}}$.

16. Translate the formula $ax + bx = (a + b)x$ into a rule for adding terms which have a common factor.

17. As in Ex. 16, translate the formula $ax - bx = (a - b)x$ into a rule, and illustrate its use in the case of $8 \cdot 4 - 3 \cdot 4$.

18. In arithmetic how do you multiply $8 + 6$ by 3? $8 - 5$ by 2? Translate the formulas $x(a + b) = ax + bx$ and $x(a - b) = ax - bx$ into rules for multiplying algebraic expressions.

19. In arithmetic we divide $6 + 8$ by 2 by changing it to $14 \div 2$. In algebra we have $\frac{a+b}{x} = \frac{a}{x} + \frac{b}{x}$ and $\frac{a-b}{x} = \frac{a}{x} - \frac{b}{x}$. Why do we not proceed as in arithmetic?

20. Translate the formulas of Ex. 19 into rules, and illustrate by dividing $12\,ab^2 + 6\,b$ and $12\,ab^2 - 6\,b$ by $6\,b$.

Perform each of the following operations and then perform the corresponding arithmetic operations which result from substituting 2 for a, 3 for b, 3 for x, and 4 for y:

21. $\frac{x}{y} + \frac{a}{b}$.

22. $\frac{x}{y} - \frac{a}{b}$.

23. $\frac{x}{y} \cdot \frac{a}{b}$.

24. $\frac{x}{y} \div \frac{a}{b}$.

CHAPTER II

REVIEW OF EQUATIONS

Linear Equation. An equation which contains the first power of an unknown quantity, and no higher power, is called a *linear equation* or an *equation of the first degree.*

Examples of several types of linear equations are as follows:

$ax - b = 0$ $\qquad$ $5x + 3 = 2x$ $\qquad$ $\frac{1}{2}x - 3 = \frac{3}{4}x$

$\frac{x}{2} + 1 = 5$ $\qquad$ $\frac{2}{x} - a = b$ $\qquad$ $\frac{x}{a} + 7 = 8b$

Steps in a Solution. In solving a linear equation of the type generally met in ordinary work there are often several steps to be taken. For example, in the equation

$$7x - 3dk^2 + 3ab^2 = k(4ax + 2d^2) - 2x$$

the different steps in the solution are as follows:

1. Remove the parentheses.

We then have $7x - 3dk^2 + 3ab^2 = 4akx + 2d^2k - 2x.$

2. Transpose the terms containing the unknown quantity x to the left side, and all other terms to the right side.

This gives us $\quad 7x - 4akx + 2x = 2d^2k - 3ab^2 + 3dk^2.$

3. Collect and combine terms.

We then have $\quad 9x - 4akx = 2d^2k - 3ab^2 + 3dk^2,$

and factoring, $\quad (9 - 4ak)x = 2d^2k - 3ab^2 + 3dk^2.$

4. Divide both sides by the coefficient of x.

This gives us $\quad x = \dfrac{2d^2k - 3ab^2 + 3dk^2}{9 - 4ak},$

which is the value of x that we wished to find. This value should be checked by substituting it in the given equation.

Fractional Linear Equations. When an equation contains fractions, the first step is generally to *clear of fractions*; that is, to make the equation integral. To do this, we multiply both sides by some convenient multiplier, which is usually the L. C. D. of the fractions.

Thus, in the equation $3\,b^2c + \frac{a+2\,b}{5\,x} = \frac{27-4\,ac}{10\,x} + 2\,d$ we may multiply by the L. C. D., $10\,x$, and have $30\,b^2cx + 2\,a + 4\,b = 27 - 4\,ac + 20\,dx$. The solution of the equation will then follow the steps given on page 21.

Exercises. Linear Equations

Solve each of the following equations for x:

1. $x - 3\,b = 5\,b$.
2. $\frac{1}{2}\,x + \frac{1}{5}\,x = 14$.
3. $\frac{1}{x-2} = \frac{3}{x-3}$.
4. $ax - bx = bc - ac$.
5. $3(4\,x - 3) + 15 = 2(x - 8)$.
6. $\frac{2\,x}{x+1} + \frac{3}{2\,x} + \frac{1}{x} - 2 = 0$.

Solve the following equations and check each result:

7. $\frac{x+2}{x-2} = \frac{10-x^2}{4-x^2} - \frac{10}{x^2-4}$.
8. $\frac{x+2}{x+3} - \frac{x+3}{x+2} = \frac{4\,x+9}{x^2+5\,x+6}$.
9. $4\,x - 18 + 16\,x - 17 - 11\,x - 10 + x = 5 + x$.

In the following cases derive formulas as specified:

10. $d = rt$; for r and t.
11. $F = \frac{9}{5}\,C + 32$; for C.
12. $i = prt$; for p, r, and t.
13. $A = p(1+r)^n$; for p.
14. $s = \frac{ar^n - a}{r-1}$; for a.
15. $L = \frac{Mt - g}{t}$; for M.
16. $l = a + (n-1)d$; for d.
17. $A = \frac{1}{2}\,h\,(b + b')$; for b.
18. $V = C\sqrt{rs}$; for C and r.
19. $s = \frac{1}{2}\,n(a + l)$; for a, n, and l.
20. $C = \frac{N\,E}{R + nr}$; for R.
21. $s = \frac{lr - a}{r - 1}$; for a, l, and r.

Simultaneous Linear Equations. When two equations in x and y are satisfied by the same set of values for x and y, the equations are said to be *simultaneous.*

For example, $x + y = 7$ and $x - y = 1$ are two such equations because they are both satisfied by the values 4 for x and 3 for y.

If a set, or system, of linear equations is to be determinate (that is, if it is to be satisfied by a single set of values for the unknowns), the number of equations in the set must be at least equal to the number of unknowns.

Elimination by Substitution. One method of solving a pair of equations is to find in one equation the value of one unknown in terms of the other, and then to substitute this value in the other equation. In this way we get rid of, or *eliminate,* one of the unknowns. For example, solve the pair of equations

$$2x + 5y = 13 \qquad (1)$$

$$7x - 4y = -19 \qquad (2)$$

Solving equation (1) for x in terms of y, we have

$$x = \tfrac{1}{2}(13 - 5y).$$

Substituting this value of x in equation (2), we then have

$$\tfrac{7}{2}(13 - 5y) - 4y = -19.$$

Solving this last equation for y, we now have

$$y = 3.$$

Substituting this value of y in equation (1), we then have

$$2x + 5 \cdot 3 = 13;$$

whence $$x = -1.$$

Check. In (1), $2(-1) + 5 \cdot 3 = -2 + 15 = 13,$

and in (2), $7(-1) - 4 \cdot 3 = -7 - 12 = -19.$

There are various other methods of eliminating one unknown, the most familiar of which is the method of elimination by addition or subtraction (see page 24). Although this latter method is readily applied in simple cases, such as are usually given in high-school examinations, in the cases that ordinarily arise in practical work the method of elimination by substitution is generally the better one to use.

Elimination by Addition and Subtraction. Another method of solving a pair of simultaneous equations depends upon eliminating one of the unknowns by using either addition or subtraction. For example, solve the pair of equations

$$x - 3y = 22 \qquad (1)$$

$$3x + 2y = 0 \qquad (2)$$

Multiplying (1) by 3, $3x - 9y = 66.$ (3)

Subtracting (3) from (2), $11y = -66;$

whence $y = -6.$

Substituting the value -6 for y in equation (1), we then have

$$x + 18 = 22;$$

whence $x = 4.$

Check. In (1), $4 - 3(-6) = 4 + 18 = 22,$

and in (2), $3 \cdot 4 + 2(-6) = 12 - 12 = 0.$

When the signs of the unknown to be eliminated are alike, as in equations (3) and (2), above, we subtract the equations, member for member; when the signs are unlike, we add the two equations.

Exercises. Simultaneous Linear Equations

Solve each of the following systems of equations:

1. $3x - 2y = 21$
   $2x + y = 21$

2. $5x - 4y = 32$
   $3x + y = 9$

3. $7x - 8y = 10$
   $5y - 6x = -16$

4. $11x + 3y = 100$
   $4x - 7y = 4$

5. $\frac{1}{x} + \frac{1}{y} = 6$
   $\frac{2}{x} + \frac{3}{y} = 16$

6. $4x + 1 = 6y$
   $5x + 1 = 7y$

7. $ax + by = c$
   $bx + cy = a$

8. $\frac{2}{3}x + \frac{3}{4}y = -\frac{7}{2}$
   $\frac{1}{4}x - \frac{2}{5}y = \frac{11}{2}$

9. $\frac{1}{2}(x + y) - \frac{1}{3}(x - y) = 8$
   $\frac{1}{3}(x + y) + \frac{1}{4}(x - y) = 11$

10. $3x - \frac{2x + y + 6}{4} = 5y$
    $\frac{3x + 2y}{6} - \frac{2y - x}{4} = -8$

In Ex. 5 it is better not to clear of fractions at first.

Ratio and Proportion. In geometry we frequently need to compare two lines; for example, if one line is 18 in. long and the other is 24 in. long, we say that the second is $\frac{1}{2}\frac{8}{4}$, or $\frac{3}{4}$, as long as the first, or that the *ratio* of the first to the second is $\frac{3}{4}$. In modern algebra a ratio is regarded simply as a fraction, and a *proportion*, which is an expression of equality between two ratios, is regarded as a fractional equation.

That is, the proportion $\frac{a}{b} = \frac{c}{d}$, which shows that the ratio $\frac{a}{b}$ is equal to the ratio $\frac{c}{d}$ and which is usually read "a is to b as c is to d," is treated as a fractional equation which can be solved, by the usual methods, for any one of the four terms a, b, c, and d.

In printing, it is sometimes convenient to express the ratio of a to b as either a/b or $a:b$, but, in writing, the form $\frac{a}{b}$ is preferred.

Mean Proportional. In solving the proportion $\frac{a}{b} = \frac{b}{c}$ we have

$$b^2 = ac;$$

whence

$$b = \pm\sqrt{ac}.$$

In such a case we speak of b as the *mean proportional* between the terms a and c.

In practical work we seldom need to use the negative square root.

Exercises. Proportion

In the proportion $\frac{a}{b} = \frac{c}{d}$ *prove that*

1. $\frac{b}{a} = \frac{d}{c}$.
2. $\frac{a}{c} = \frac{b}{d}$.
3. $\frac{a^2}{b^2} = \frac{c^2}{d^2}$.
4. $\frac{a^n}{b^n} = \frac{c^n}{d^n}$.
5. $-\frac{a}{b} = -\frac{c}{d}$.
6. $\frac{a}{b} + 1 = \frac{c}{d} + 1$.
7. $\frac{a+b}{b} = \frac{c+d}{d}$.
8. $\frac{a-b}{b} = \frac{c-d}{d}$.
9. $\frac{a-c}{c} = \frac{b-d}{d}$.
10. $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.
11. $\frac{ax}{by} = \frac{cx}{dy}$.
12. $\frac{ax+by}{cx+dy} = \frac{b}{d}$.
13. $\frac{5a+3c}{5a-3c} = \frac{5b+3d}{5b-3d}$.
14. $\frac{ma^2+nc^2}{mb^2+nd^2} = \frac{a^2}{b^2}$.
15. $\frac{ab+cd}{ab-cd} = \frac{a^2+c^2}{a^2-c^2}$.

Find the mean proportional between

16. 1 and 4. 18. 6 and 24. 20. 3 and 3. 22. m and $9m$.

17. 2 and 8. 19. 5 and 20. 21. 3 and 8. 23. n^2 and $4n^4$.

24. In the proportion $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}$, prove that $\frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}$.

If we let each ratio equal r, we have $a = br$, $c = dr$, $e = fr$, and $g = hr$. Then $a + c + e + g = br + dr + fr + hr = (b + d + f + h)r$, which can be divided by $b + d + f + h$.

25. Solve the equation $\frac{V}{V_0}=\frac{P_0 t}{P t_0}$ for P_0; for t; for t_0.

26. In Ex. 25, V_0 represents the initial volume of a gas at an initial temperature t_0 (on the absolute scale) under an initial pressure P_0, and V represents the volume at the temperature t and the pressure P. In such a case we say that the volume of the gas is directly proportional to the absolute temperature and inversely proportional to the pressure. If at $-73°$ C. the volume of a gas is 8 cu. ft. under a pressure of 14 lb., what is its volume at 27° C. under a pressure of 20 lb.?

Writing the equation of Ex. 25 as $\frac{V}{V_0}=\frac{t}{t_0}\cdot\frac{P_0}{P}$, we see that the order of the V's and the t's is the same, but the order of the P's is the inverse of that of the V's. That is, the first is a direct proportion in which an increase in t causes an increase in V; while the second is an inverse proportion in which an increase in P causes a decrease in V.

The expression $-73°$ C. means 73° below zero on the centigrade scale. On the absolute scale, 0° corresponds to $-273°$ C; that is, add 273 to the centigrade reading in order to convert it to absolute temperature.

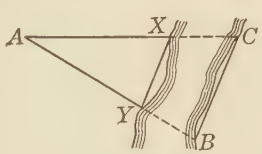
27. In Ex. 26, what is the pressure necessary at 27° C. in order to keep the volume of the gas the same as it was at $-73°$ C. under a pressure of 14 lb.?

28. The areas of similar plane figures are proportional to the squares of their corresponding dimensions. What is the area of a circle with a radius $2\frac{1}{2}$ times as long as that of a circle which contains 100 sq. ft.?

29. The radius of a circle containing 1000 sq. ft. is how many times as long as that of one which contains 100 sq. ft.?

30. The volumes of similar solids are proportional to the cubes of their corresponding dimensions. What is the diameter of a sphere which has a volume twice that of a sphere with a radius of $3\frac{1}{2}$ in.?

31. The corresponding sides of similar triangles are always in proportion. In this figure in which AXC is a straight line and XY is parallel to BC, find the width XC of the river in terms of AX, BC, and XY, the lengths of which can be easily measured.



32. On a balanced lever the forces are inversely proportional to their distances from the fulcrum. What force must be applied to the free end of a 42-inch crowbar in order to balance the force exerted by a 300-pound stone resting on the other end at a distance of 7 in. from the fulcrum?

33. The capacities of silos of the same height are proportional to the squares of the inside diameters. Find, to the nearest 1 in., the inside diameter of a silo having twice the capacity of one with the same height and a diameter of 14 ft.

34. If water is flowing at a uniform rate through a pipe leading to a tank, and if W represents the weight of water in the tank after t minutes of flow, what does it mean to say that $\frac{W_1}{W_2} = \frac{t_1}{t_2}$? What is the value of t_2 in terms of W_1, W_2, and t_1?

35. From the formulas $A_1 = \frac{1}{4}\pi d_1^2$ and $d_2^2 = \frac{4A_2}{\pi}$ find the ratio of d_1 to d_2 in terms of A_1 and A_2.

36. Under an equal head (pressure) the amount of water discharged in a given time from two pipes is, for practical purposes, directly proportional to the squares of the diameters and inversely proportional to the square roots of the lengths. During the time that a $1\frac{1}{2}$-inch pipe 2000 ft. long discharges 3000 gal., what will be the discharge under the same head from a 2-inch pipe 3000 ft. long?

Quadratic Equation. When reduced to its simplest form, an equation which contains the second power of an unknown quantity, but no higher power, is called a *quadratic equation.*

For example, $a^2x^2 = 9$ and $3x^2 - 2x = 4$ are quadratic equations. A quadratic equation is often called simply a *quadratic.*

The equation $ax^2 + bx + c = 0$ is called the *standard form* or *general form* of a quadratic equation. If a quadratic equation contains both the first and second powers of the unknown quantity, it is called a *complete quadratic*; if only the second power, it is called an *incomplete quadratic.*

For example, $x^2 + 9x = 6$ is a complete quadratic, while $x^2 - 9 = 0$ is an incomplete quadratic.

Solving an Incomplete Quadratic. An incomplete quadratic equation has the form

$$ax^2 + 0 \cdot bx + c = 0,$$

and hence such an equation can always be reduced to the form

$$ax^2 = -c.$$

This equation can then be solved by simply dividing by a and taking the square root; that is, we have

$$x = \pm\sqrt{\frac{-c}{a}}.$$

For example, in solving the equation $3x^2 - 12 = 0$ we have

$$x = \pm\sqrt{\frac{-(-12)}{3}} = \pm\sqrt{4} = \pm 2.$$

Similarly, the equation $7x^2 + 8x + 3 = 8x + 38$ can be reduced to

$$7x^2 - 35 = 0;$$

whence

$$x = \pm\sqrt{\frac{-(-35)}{7}} = \pm\sqrt{5}.$$

Each of these solutions should be checked by substituting in the original equation; thus, in $3x^2 - 12 = 0$, we have

$$3(\pm 2)^2 - 12 = 12 - 12 = 0.$$

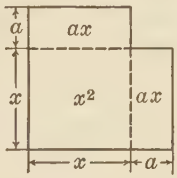
For the left side of the equation $7x^2 + 8x + 3 = 8x + 38$ we have

$$7(\pm\sqrt{5})^2 + 8(\pm\sqrt{5}) + 3 = 35 \pm 8\sqrt{5} + 3 = 38 \pm 8\sqrt{5},$$

and for the right, $8(\pm\sqrt{5}) + 38 = 38 \pm 8\sqrt{5}$.

Completing the Square. In practical work, quadratic equations are generally solved by a method which depends upon a process known as *completing the square.*

This process may be illustrated geometrically by completing this figure so that it will form a square. In order to do so we must obviously add a small square with the side a and the area a^2. In the figure as given, the area is $x^2 + 2\,ax$; after adding the small square, the area becomes $x^2 + 2\,ax + a^2$. This is, of course, the square of $x + a$, and hence it may be written as $(x + a)^2$.



That is, in order to complete the square of an algebraic expression of the form $x^2 + 2\,ax$,

Add the square of half the coefficient of x.

For example, in the case of $x^2 - 4\,x$ we complete the square by adding 4, the square of -2.

Exercises. Completing the Square

Complete the square of each of the following expressions:

1. $x^2 + 2\,xy$.
2. $x^2 - 2\,xy$.
3. $x^2 + xy$.
4. $x^2 - xy$.
5. $x^2 + 4\,x$.
6. $x^2 + 8\,x$.
7. $x^2 - 10\,x$.
8. $x^2 + 3\,x$.
9. $x^2 - 5\,x$.
10. $x^2 + 13\,x$.
11. $x^2 - \frac{1}{2}\,x$.
12. $x^2 + \frac{1}{3}\,x$.
13. $x^2 - \frac{1}{8}\,x$.
14. $x^2 + \frac{2}{3}\,x$.
15. $x^2 - \frac{5}{6}\,x$.

16. In the equation $x^2 - 6\,x = 0$, what must you add to both sides of the equation so that the left side will be the square of a binomial? Of what binomial will it be the square? What will be the square root of the left side? of the right side? Since these square roots will be equal, how can you solve the resulting equation? What are the values of x?

17. As in Ex. 16, complete the square of the first member of the equation $x^2 + 10\,x = 0$ by adding a certain number to each side, and then solve the resulting equation.

Solution by Completing the Square. We are now ready to apply the process of completing the square to the solution of a quadratic equation. For example, let us consider the equation

$$3x^2 - 5x = 37.$$

Dividing by 3, $x^2 - \frac{5}{3}x = \frac{37}{3}$.

Completing the square by adding to both sides the square of half the coefficient of x, or $(\frac{5}{6})^2$, we have

$$x^2 - \tfrac{5}{3}x + (\tfrac{5}{6})^2 = \tfrac{37}{3} + \tfrac{25}{36} = \tfrac{469}{36}.$$

Taking the square root of both sides of this equation, we then have

$$x - \tfrac{5}{6} = \pm\sqrt{\tfrac{469}{36}} = \pm\tfrac{1}{6}\sqrt{469};$$

whence $$x = \tfrac{5}{6} \pm \tfrac{1}{6}\sqrt{469},$$

or $$x = \frac{5 \pm \sqrt{469}}{6}.$$

From a table of square roots, such as is given on pages 298–301, we can find that $\sqrt{469} = 21.66$, to the nearest 0.01, and hence the approximate values of x are 4.44 or -2.78. Unless we are solving a problem in which measurements of some kind are involved, we usually leave the value of x in the form $x = \frac{5 \pm \sqrt{469}}{6}$, as given above.

Exercises. Solution by Completing the Square

Solve the following equations by completing the square:

1. $x^2 - 12x = -11$.
2. $x^2 + 6x = 27$.
3. $x^2 + 5x = 50$.
4. $x^2 + 7x = 120$.
5. $x^2 + x = 132$.
6. $2x^2 + 7x - 39 = 0$.
7. $3x^2 - 5x + 2 = 0$.
8. $x^2 - ax + b = 0$.
9. $a^2x^2 - abx + c = 0$.
10. $ax^2 - bx + c = 0$.

In each of the following equations find the value of x in terms of y:

11. $x^2 - 6xy - 5y^2 = 0$.
12. $x^2 - 4y^2 = 3xy$.
13. $x^2 - 2xy = 24y^2$.
14. $x^2 - 11xy + 28y^2 = 0$.
15. $2x^2 + 15xy = 8y^2$.
16. $x^2y + 85y^3 = 22xy^2$.
17. $b^2x^2 + 2bxy - 8y^2 = 0$.
18. $3x^2 - 2x = 4y^2 + 7y$.

The Quadratic Formula. In a case involving an equation like

$$0.32\,x^2 - 1.73\,x - 2.8 = 0,$$

a type that often arises in practical problems, it is not a simple matter to solve the equation by completing the square. We shall therefore review another and more general method.

From the general form of the quadratic equation,

$$ax^2 + bx + c = 0,$$

we have

$$ax^2 + bx = -c,$$

and, dividing by a,

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Completing the square of this equation, we then have

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2};$$

whence

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a},$$

and

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This value of x may be used as a formula for finding the roots of any quadratic equation.

For example, in a simple case like $3x^2 - 5x - 12 = 0$, we see that $a = 3$, $b = -5$, and $c = -12$, and hence we have

$$x = \frac{5 \pm \sqrt{25 + 144}}{6} = \frac{5 \pm 13}{6} = 3 \text{ or } -1\tfrac{1}{3}.$$

Applying the formula to the equation at the top of the page, we have

$$x = \frac{1.73 \pm \sqrt{1.73^2 - 4 \times 0.32 \times (-2.8)}}{2 \times 0.32}$$

$$= \frac{1.73 \pm \sqrt{2.9929 + 3.584}}{0.64}$$

$$= \frac{1.73 \pm 2.564}{0.64}$$

$$= 6.71 \text{ or } -1.30, \text{ each to the nearest } 0.01.$$

The accuracy to which the result should be given would be determined by the conditions in a practical case. In such work it frequently happens that the negative root has no significance and may therefore be discarded.

Exercises. Solution by Formula

Solve the following equations by using the formula:

1. $x^2 - x - 90 = 0$.
2. $x^2 + x - 90 = 0$.
3. $x^2 + 19\,x + 90 = 0$.
4. $x^2 - 19\,x + 90 = 0$.
5. $x^2 + 3\,x - 70 = 0$.
6. $x^2 - 3\,x - 70 = 0$.
7. $x^2 + 22\,x + 121 = 0$.
8. $x^2 - 22\,x + 121 = 0$.
9. $x^2 + 12\,x - 13 = 0$.
10. $x^2 + 3\,x - 130 = 0$.
11. $6\,x^2 + x - 1 = 0$.
12. $6\,x^2 - x - 1 = 0$.
13. $8\,x^2 + 8\,x = 6$.
14. $15\,x^2 + 32\,x = 7$.
15. $x^2 - \frac{1}{12}\,x - \frac{1}{2} = 0$.
16. $x^2 + 2\frac{1}{6}\,x - 10\frac{1}{2} = 0$.
17. $7\,x^2 + 0.6\,x - 0.4 = 0$.
18. $9\,x^2 + 1\frac{4}{7}\,x - \frac{2}{7} = 0$.
19. $\frac{1}{144}\,x^2 - \frac{1}{6}\,x + 1 = 0$.
20. $x^2 - (\sqrt{3} + \sqrt{2})x + \sqrt{6} = 0$.

21. Solve the equation $x^4 - x^2 = 12$ as a quadratic in x^2. After finding the value of x^2, find the value of x and check by substituting in the original equation.

22. Regarding the equation $x^2 + 3\,x + \sqrt{x^2 + 3\,x} - 6 = 0$ as a quadratic in $\sqrt{x^2 + 3\,x}$, solve for $\sqrt{x^2 + 3\,x}$; then square both sides and solve the resulting equation for x. Check each root by substituting it in the original equation.

23. The roots of $x^2 - 4\,x + 1 = 0$ are either $3 \pm \sqrt{2}$ or $2 \pm \sqrt{3}$. Without solving, state which they are.

24. Solve the equation $x^2 - 2\,x - 4\,\sqrt{x^2 - 2\,x - 2} = 7$ and check each root by substituting it in the original equation.

First write the equation in the form $x^2 - 2\,x - 2 - 4\,\sqrt{x^2 - 2\,x - 2} - 5 = 0$.

25. Solve the equation $x^2 - 3\,x - 2\,\sqrt{x^2 - 3\,x + 4} = -4$.

Solve each of the following equations:

26. $x^2 + 10\,x = 75$.
27. $\dfrac{(x+2)\,(x-2)}{5} = \left(\dfrac{3}{x}\right)^2$.
28. $24\,x^2 - 97\,x - 70 = 0$.
29. $\dfrac{a}{x-a} + \dfrac{b}{x-b} = \dfrac{2\,c}{x-c}$.

Exercises. Test on Chapter II

Solve each of the following linear equations:

1. $7x = 2.24$.
2. $\frac{1}{7}x = 2.24$.
3. $\frac{3}{5}x = \frac{1}{2}x + 2\frac{1}{2}$.
4. $x - 3(2 - 5x) = 32$.
5. $3(x + 15) + 5 = 2(2x - 3)$.
6. $2(x^2 - x) = 2x^2 - 16$.

Solve each of the following fractional equations:

7. $\frac{21}{x} = 7$.
8. $\frac{9}{x} = 18$.
9. $\frac{5-x}{x+3} = \frac{2-x}{4+x}$.
10. $\frac{x+3}{x+2} = \frac{x+1}{x+3}$.
11. $\frac{x-4}{x-12} = \frac{x-8}{x-13}$.
12. $\frac{2x+3}{5-2x} = \frac{3x+4}{5-3x}$.

Solve each of the following sets of linear equations:

13. $3x + 2y = 7$
    $2x + 3y = 8$
14. $3x + 4y = 11$
    $5x - y = 3$
15. $x + y = a$
    $x - y = b$

Solve each of the following quadratic equations:

16. $x^2 - 6x - 7 = 0$.
17. $x^2 + 8x + 7 = 0$.
18. $x^2 + 6x - 7 = 0$.
19. $x^2 + 7x - 8 = 0$.
20. $x^2 + 9x + 8 = 0$.
21. $x^2 - 7x + 8 = 0$.
22. $x^2 + 6x - 27 = 0$.
23. $x^2 - 6x - 27 = 0$.
24. $x^2 + 12x + 27 = 0$.
25. $x^2 + 12x + 36 = 0$.
26. $x^2 - x = 132$.
27. $x^2 + 23x = -132$.
28. $x^2 + 11x = 26$.
29. $x^2 = 11x + 26$.
30. $x^2 = 13x + 14$.
31. $6 - x = x^2$.
32. $13x + 140 = x^2$.
33. $238 = 3x + x^2$.
34. $6x^2 - 7 = 11x$.
35. $x^2 + x - 1 = 0$.
36. $x^2 + 3x = 70$.
37. $x^2 + x = 156$.

38. If the length of a rectangular field which contains exactly 1 A. (acre) is 2 rd. more than its width, what, to the nearest 0.1 rd., are the dimensions of the field?

Exercises. General Review

1. A man bought a farm for \$24,000 and then sold all but 80 A. of it for the same amount. If he received \$50 per acre more than the purchase price, how many acres did he buy?

2. In the equation $t^2 + 517\,t - e = 0$, t expresses the number of degrees by which the boiling point of water (100°C.) is lowered at an elevation of e feet above sea level. At an elevation of 8000 ft., at what temperature will water boil?

3. A farmer has a 5-acre field which is twice as long as it is wide. If he should erect a second fence at a uniform distance of $16\frac{1}{2}$ ft. inside the fence around the field, by how much would he reduce the area of the field?

4. The effective area E of a chimney is given by the formula $E = A - 0.6\sqrt{A}$, where A is the measured area. Find A when $E = 21$; when $E = 7.6$.

5. A sheet of tin that is twice as long as it is wide is made into an open box containing 192 cu. in. by cutting a 2-inch square out of each corner and turning up the sides. What were the dimensions of the original sheet of tin?

6. By how much must the radius of a circle 28 in. in diameter be reduced in order to decrease the area 156 sq. in.?

In all such cases take $\frac{22}{7}$ as the value of π.

7. If a city lot with an area of 9000 sq. ft. has a depth of 30 ft. more than twice its frontage, what are its dimensions?

8. A man took $10\frac{1}{2}$ hr. to make the round trip to a city 90 mi. away. If on the return trip he increased his average speed by 5 mi./hr., what was his average speed each way?

The symbol "mi./hr." is read "miles per hour."

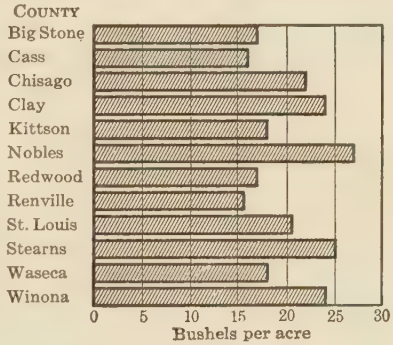
9. If the larger of two given numbers is divided by the smaller, the quotient is 4 and the remainder is 1. If the smaller is divided by the larger, the quotient is 0.24 and the remainder is 0.02. What are the numbers?

CHAPTER III

GRAPHS AND FUNCTIONS

Graphic Representation. In the newspapers or magazines of the day statistical material of all sorts is very often presented by means of diagrams or pictures. Although these graphic representations do not supply any more information than can be obtained from numbers arranged in columns in a table, they generally set forth the important facts much more clearly and forcefully.

Bar Graph. Taken together, the bars in the figure at the right present a graphic story of the wheat production in a recent year in twelve different counties in Minnesota. A diagram like this, which is made up of a system of bars or lines drawn to scale, is called a *bar graph.*



Wheat Production in Minnesota

In drawing such a graph, a definite length is taken to represent the unit of production. In this case, $\frac{1}{4}$ in. is taken to represent 5 bu. per acre; and hence a production of 15 bu. per acre is represented by a bar $\frac{3}{4}$ in. long. The lengths of the bars representing productions like 17 bu. per acre or 23 bu. per acre are easily estimated as fifths of the $\frac{1}{4}$-inch unit.

In such graphs the bars are often placed vertically instead of horizontally, depending upon which is the more convenient.

Although a treatment of graphs is rapidly becoming an integral part of any course in elementary algebra, the preparation of students in this topic will probably be found to vary widely. The amount of time to be devoted to this review chapter will therefore depend to a large extent upon the individual differences of classes in this respect.

Drawing Bar Graphs. The following suggestions will be found helpful in drawing bar graphs:

1. *Decide upon the most convenient unit.*

It is usually best to take 1 in., or some convenient fraction of 1 in., to represent a definite number of units of measure, such as 100 bu., 10 lb., 50 A., and so on.

In all kinds of scientific work the metric system is now generally used, and the important subdivisions of the meter, the centimeter (1 cm.) and the millimeter (1 mm.), are often convenient units in drawing graphs.

The scale of the graph should be suited to the size of the paper upon which the graph is to be drawn. A little experimenting is often necessary in order to avoid having the longest bar too long for the sheet or the shortest bar so short that it cannot be measured conveniently.

2. *Mark the zero point and give any other facts needed in interpreting the scale used.*

In the graph on page 35 the vertical lines are so marked that the number of bushels per acre can be read directly from the graph.

3. *Make the bars of uniform width, so that the reader will compare them simply by their lengths.*

In the graph on page 35 the bars are slightly separated, and the student should follow this plan in the exercises on pages 37 and 39. Later we shall use graphs in which it is better not to do this.

4. *Either write a brief title or label the graph in such a way that the purpose of the graph is evident.*

In the case on page 35 the title is "Wheat Production in Minnesota," and the headings "County" and "Bushels per acre" complete the story.

5. *Be as precise in making measurements as the instruments used permit.*

A carelessly drawn graph not only indicates slovenly habits of work, but it will not tell a true story.

The student will find it convenient to have a 6-inch boxwood scale, which can be carried readily in his pocket. This type of scale has a sharp edge that will lie close to the paper and the graduations are carefully made. The lengths of the bars can then be readily pricked off on the paper with a pin point or a needle point.

If only an ordinary ruler is available, short lengths can be transferred to the paper by means of a pair of dividers.

Exercises. Bar Graphs

1. The population, to the nearest hundred, per square mile in each of six states was recently as follows: Rhode Island, 600; Massachusetts, 500; New Jersey, 400; Connecticut, 300; New York, 200; Pennsylvania, 200. Represent these statistics by a bar graph, and state the probable reasons for the variations.

2. In a recent year the value, to the nearest billion, of the manufactured products in each of nine manufacturing states was as follows: Massachusetts, $4,000,000,000; New York, $9,000,000,000; Pennsylvania, $7,000,000,000; Ohio, $5,000,000,000; Illinois, $5,000,000,000; Michigan, $3,000,000,000; New Jersey, $4,000,000,000; Indiana, $2,000,000,000; Wisconsin, $2,000,000,000. Represent these statistics by a bar graph.

3. The average heights in feet of American boys at two to twenty years of age are given in the following tabulation:

Age in years	2	4	6	8	10	12	14	16	18	20
Height in feet	1.6	2.6	3.0	3.5	4.0	4.8	5.2	5.5	5.6	5.7

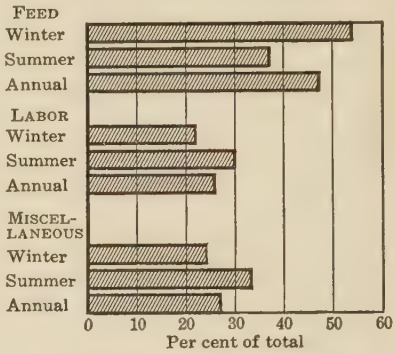
Represent these statistics by a bar graph. At about what age does the average boy grow most rapidly? From the graph make a statement about the height of an average boy 19 yr. old.

4. As in Ex. 3, draw and consider the graph of the following statistics relating to the average heights of American girls:

Age in years	2	4	6	8	10	12	14	16	18	20
Height in feet	1.6	2.6	3.0	3.5	3.7	4.5	4.8	5.1	5.3	5.4

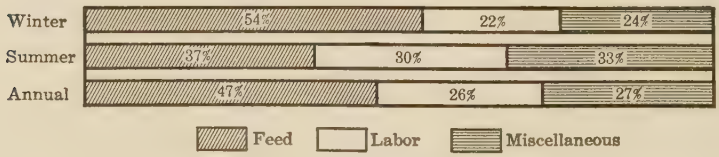
5. To the nearest 100 mi., the lengths of five of our longest rivers are as follows: Arkansas, 2000 mi.; Platte, 1300 mi.; Rio Grande, 1800 mi.; Missouri-Mississippi, 4200 mi.; Columbia, 1400 mi. Rearranging the names in order, beginning with the shortest, represent these statistics by means of a graph with horizontal bars.

Comparative Bar Graphs. Thus far, each graph that we have considered has represented only a single group of statistics. It is frequently necessary, however, to compare and draw conclusions from two or more related groups of statistics. For example, the graph at the right was drawn to show for a large milk-producing region the per cents of the total cost of milk production represented by feed, labor, and miscellaneous expenses respectively. The bars not only show the per cents expended for each purpose during the year, but they also show how the per cents varied during the summer and winter.



Cost of Milk Production

In comparing statistics of this sort it is frequently more convenient to use bars that are divided proportionally to show the distribution of the items of expense, as in the figure below, which represents the same statistics as the above graph.



Cost of Milk Production

The third bar of this graph shows at a glance that the cost of feed was nearly half the total annual expenses and that labor and miscellaneous expenses were each about a fourth. The graph also shows that although the relative cost of feed increased during the winter the relation of labor expense to miscellaneous was practically the same throughout the year.

Drawing conclusions, like those given above, from a graph is generally called *interpreting the graph.*

Exercises. Comparative Bar Graphs

1. Combine in one diagram the graphs drawn for Exs. 3 and 4 on page 37, using shaded bars for the boys' heights and white bars for the girls' heights and arranging them alternately. Write a brief statement about the comparative heights of average boys and girls at these ages.

2. In another locality in the same state and during the same year to which the comparative bar graphs on page 38 referred, the statistics of the cost of milk production were as follows:

Season	Feed	Labor	Miscellaneous
Winter	60.3 %	18.6 %	21.1 %
Summer	36.0 %	30.0 %	34.0 %
Annual	51.0 %	23.0 %	26.0 %

Represent these statistics by a bar graph similar to the second one on page 38. State whether or not this graph shows the same tendencies as the graphs on page 38.

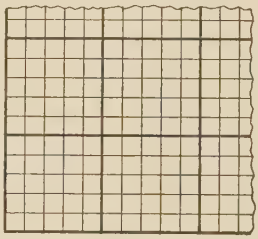
3. The following table shows both the approximate value and the cost of production for a double five-year rotation of crops in one of the North Central States. As in Ex. 2, represent these statistics by a bar graph similar to the second one on page 38 and determine which crop yields the greatest amount of profit:

Year	Crop	Cash Value	Cost
1	Pasture	$2.89	$5.75
2	Oats	30.09	13.64
3	Corn	48.00	21.55
4	Wheat	52.23	15.01
5	Meadow	90.63	13.41
6	Pasture	14.25	10.02
7	Oats	34.10	26.01
8	Corn	23.78	28.41
9	Wheat	29.14	14.51
10	Meadow	7.62	11.84
Average		$33.27	$16.02

Squared Paper. Usually graphs can be most easily drawn on *squared paper*; that is, on paper which is ruled in squares.

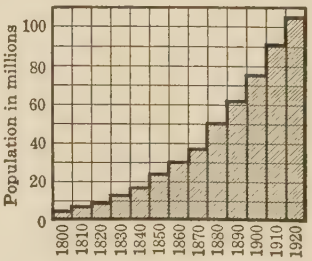
Such paper is known also as *coordinate paper*, *cross-section paper*, or *cross-ruled paper*, and can be bought in sheets or pads at any stationer's. The student should secure or rule some of this paper, as we shall need to use it often in our work.

Squared paper is generally ruled in inches and the common fractions of an inch, in inches and decimal fractions of an inch (as shown in the figure at the right), or in units of the metric system. Engineers and architects more often use the first two kinds, while scientists commonly prefer the third kind.



In our work either the decimally-ruled or the metric-ruled paper will be most convenient.

Bar Graphs on Squared Paper. When the statistics to be compared are taken at various intervals, as in the case of the census reports made every 10 yr., it is customary to arrange the bars so that the successive periods follow in natural order from left to right. The accompanying graph, which shows how the population of the United States has increased at 10-year intervals from 1800–1920, illustrates how convenient squared paper is in drawing graphs.



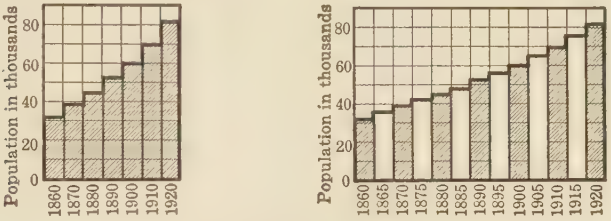
Population of the United States, 1800–1920

In this case the student should notice that the figures opposite the horizontal lines represent millions; that is, 20 means 20,000,000. When all the figures in a set of statistics represent millions or billions, and so on, it is easier to compare the numbers if we use only the figures representing the number of millions or billions.

The student should also notice that in this case the bars have not been separated, as in the case of the graph on page 35. By not doing this, the increases in the population are more graphically shown by means of the steps in the flight of stairs suggested by the outline of the graph.

Interpolation. In a graph representing a census taken every 10 yr. we can usually make a fairly close approximation of the population at the intermediate 5-year intervals or even at yearly intervals.

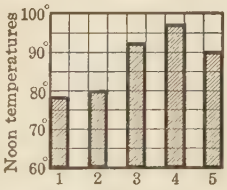
Suppose, for example, that in a city the number of inhabitants at each 10-year interval from 1860 to 1920 was 32,000, 38,000, 45,000, 52,000, 60,000, 70,000, and 82,000 respectively. From these figures we may draw the graph shown below at the left.



Population of a City

Since the graph shows that the population increased at practically a uniform rate, we may assume that the points halfway between the tops of two successive bars represent the population at the intermediate 5-year intervals. We can then insert bars, the white ones in the diagram at the right, to represent these years. When we determine intermediate figures in this way, the operation is called *interpolation.*

Similarly, to interpolate for every 2 yr., we should take 0.2 of the difference between two 10-year bars for the first interpolated bar, 0.4 of the difference for the second bar, 0.6 for the third, and 0.8 for the fourth.



Noon Temperatures

Of course, we must use common sense to determine when it is proper to interpolate and when it is not. For example, in this graph of the noon temperatures for five successive days in July, it is not proper to take the halfway points as the midnight temperatures, because these temperatures might easily be as low as 50° or 60°.

Exercises. Drawing Graphs

1. The value, in hundreds of millions of dollars, of the wheat crop of the United States in each of eight years, starting with 1915, was as follows: 9, 10, 13, 19, 21, 12, 7, 7. Represent these statistics by a bar graph and state one cause for the sudden rise for three of the years.

2. In the same years as in Ex. 1 the value of the corn crop, in hundreds of millions of dollars, was as follows: 17, 23, 39, 34, 35, 22, 13, 18. Represent these statistics by a bar graph and state one cause for the sudden rise for three of the years.

3. Combine the graphs of Exs. 1 and 2 in a single diagram and from it decide whether or not the rise in value for the same three years was as marked for wheat as for corn.

4. The population of the United States per square mile on each of seven census dates was approximately as follows:

Year	1800	1820	1840	1860	1880	1900	1920
Population	6	5.5	10	11	17	26	36

Represent these statistics by a bar graph. From your study of history try to explain the variations.

5. In Ex. 4 find by interpolation the population per square mile for each 10-year interval from 1800 to 1920, and draw a graph showing the interpolated bars.

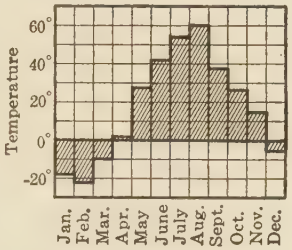
6. The following table shows the horse power developed by the engines in driving a large steamship at five different speeds:

Speed in miles per hour . . .	4	8	12	16	20
Horse power developed . . .	200	1200	3300	8200	16,000

Draw a graph of these statistics and find the power needed to drive the ship at 10 mi./hr.; at 18 mi./hr. Write a statement showing how the horse power required is related to the speed.

Negative-Number Graph. A graph furnishes a simple and effective method of representing negative numbers. For example, suppose that in one of our northern states the lowest recorded temperatures during each month in a recent year were as follows:

Month	Temperature	Month	Temperature
Jan. . .	− 18°	July . .	54°
Feb. . .	− 22°	Aug. . .	60°
Mar. . .	− 10°	Sept. . .	38°
Apr. . .	2°	Oct. . .	27°
May . .	28°	Nov. . .	15°
June . .	42°	Dec. . .	− 5°



Lowest Temperatures

In representing these statistics graphically we simply let the bars for the negative temperatures extend in the opposite direction from those which indicate positive temperatures. The variations in the temperatures are much more obvious in the graph than they are in the table.

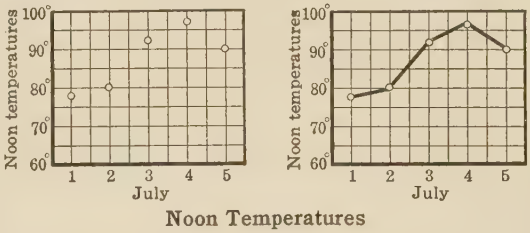
Exercises. Graphs with Negative Numbers

1. In ten consecutive years the departures in inches from the mean seasonal rainfall for the St. Paul district were as follows: − 3.15, + 3.02, − 3.26, − 2.84, + 2.42, + 2.61, − 3.07, − 2.95, − 2.27, − 7.52. Draw a bar graph to represent these statistics and write a brief statement of the facts revealed.

2. Under average conditions it is estimated that 1 bu. of corn produces 10 lb. of pork. Draw a bar graph to represent the profit and loss to the farmer for each 100 lb. of pork produced in seven consecutive years in which the prices per hundred-weight received from the sale of hogs and the prices per bushel of corn were as shown in the following table:

Price of hogs .	$7.60	$6.69	$7.61	$12.10	$15.78	$16.60	$13.43
Price of corn .	0.72	0.72	1.21	1.47	1.52	1.51	0.64

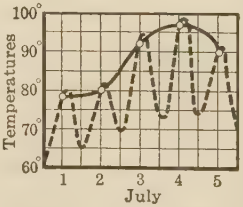
Broken-Line Graphs. In the temperature graph on page 41 we were interested chiefly in the changes in the tops of the rectangles. Instead of using a bar graph, we may first draw a horizontal line on which to mark off the days and then a vertical line for the temperatures. We may then indicate each noon temperature by a point, as in the first of these figures, and to help the eye to follow the changes in the temperature, we may connect the points, as in the second figure, by a series of short straight lines. Such a series of lines is usually called a *broken line,* and hence the graph is called a *broken-line graph.*



Noon Temperatures

Curve-Line Graphs. Instead of using a broken line, we may under certain conditions use a curve line. Thus, if we wish the above graph to show only the range of noon temperatures, we may use the solid curve shown in this figure.

If we had sufficient data, we might find that the temperature during each 24 hr. in the period of 5 da. varied as shown by the dotted curve. This dotted curve shows why it is not proper, as stated on page 41, to interpolate the midnight temperatures from the solid curve.



Noon Temperatures

Since a rapid increase is represented by a sharp upward rise of the curve, or a slow decrease by a gradual drop, the curve-line graph is especially useful in cases where it is desired to show variations in temperatures, rainfall, crop yields, commodity prices, and the like.

The subject of statistics and statistical graphs, with special applications to agricultural problems, will be considered in detail in Chapter VI. In this preliminary review it is desirable to limit the work to the more simple applications that are of general interest.

Exercises. Curve-Line Graphs

1. Draw a curve-line graph of the temperatures on a day when the thermometer registered as follows: 6 A.M., 50°; 8 A.M., 60°; 10 A.M., 62°; noon, 75°; 2 P.M., 75°; 4 P.M., 60°; 6 P.M., 50°.

2. A steam gauge showed the following pressures in pounds per square inch: Oct. 1, 8 A.M., 140; 9 A.M., 142; 10 A.M., 142. Locate these points and draw the pressure curve.

3. Other readings of the gauge in Ex. 2 were as follows: Oct. 1, 4 P.M., 140; 5 P.M., 138; Oct. 2, 8 A.M., 138; 9 A.M., 142. Locate these points on the graph in Ex. 2. Is it proper to extend the curve in Ex. 2 to these four points?

4. The average weights of American children between the ages of seven and twelve years are given in the following table:

Age in years	7	8	9	10	11	12
Weight in pounds	47	52	57	62	69	76

Represent these facts by a curve-line graph and from it find the average weight of a child $8\frac{1}{2}$ yr. old.

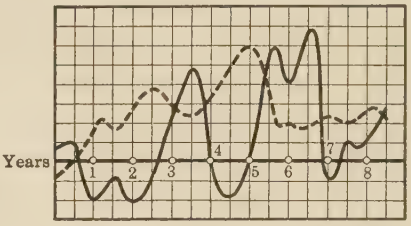
5. The following table shows the number of years a man may expect to live (expectancy of life) at different ages:

Age	10	15	20	25	30	35	40	45	50
Expectancy of life	49	46	42	39	35	32	28	25	21

Represent these statistics by a curve-line graph and from it find a man's expectancy of life at the age of 23; of 28; of 42; of 48.

6. For the last ten years the annual rainfall in inches in a certain river valley was as follows: 22.8, 20.1, 32.2, 24.5, 27.3, 28.4, 20.9, 24.6, 29.7, and 23.9. Represent these statistics by a curve-line graph. Is it proper to make a forecast from the graph concerning the rainfall for the next year?

Comparative Curve-Line Graphs. Just as on page 38 we found it convenient to use two or more bar graphs when comparing related sets of statistics, so we shall find it of value to learn how to apply similar methods to two or more curve-line graphs.



Relation of Car Supply to Prices

The solid curve in this figure shows the supply of freight cars in this country during a period of about nine years. The dotted curve shows the general trend of prices of standard commodities, like food and clothing. In Ex. 1, below, we shall see how conclusions can be drawn from this graph.

The line marked "Years" represents the average supply of freight cars. When the solid curve drops below that line, it indicates a shortage of cars; when it rises above, it indicates a surplus.

Exercises. Comparison of Graphs

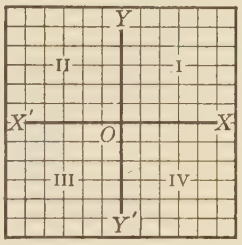
1. From the above graph write a brief statement, with reasons, of the relation which appears to exist between the rise and fall of prices and the shortage or surplus of cars.

2. Draw graphs to show the following per cents made by two students in twelve successive tests in mathematics: Ray Clark, 60, 70, 75, 81, 85, 90, 87, 90, 89, 92, 95, 97: Robert Dunton, 60, 65, 50, 70, 75, 70, 81, 87, 85, 89, 90, 92. Compare the two curves and state your conclusions.

3. Draw graphs to represent the following average weights, in pounds, of boys and girls at thirteen different ages:

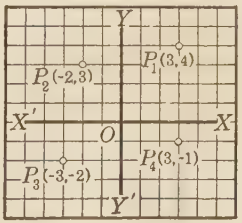
Age	6	7	8	9	10	11	12	13	14	15	16	17	18
Boys	43	48	52	58	63	69	75	84	94	106	117	126	135
Girls	42	46	51	56	61	68	76	86	97	106	112	115	117

Locating Points in a Plane. We may now apply the principles used in drawing a graph to the location of points in a plane. We simply take two lines, like $X'X$ and $Y'Y$ in this figure, as the base lines from which to describe the position of the points. These two lines, which are generally taken perpendicular to each other, are called the *axes*. The line $X'X$ is called the *x axis*, and the line $Y'Y$ is called the *y axis*. The point of intersection of the axes is called the *origin*, and is usually designated by O.



We also speak of the four sections into which the plane is divided by the axes as the four *quadrants*. We number these quadrants I to IV, as shown, taking the quadrant formed by the lines OX and OY as I and proceeding in counterclockwise order; that is, opposite to the motion of the hands of a clock.

Coordinates. The distance of a point from the y axis is called the *abscissa* of the point, and the distance from the x axis is called the *ordinate*. Together, the abscissa and the ordinate of a point are called the *coordinates* of the point.



Thus, in this figure the abscissa of the point P_1 is 3 and the ordinate is 4.

Abscissas to the right of the y axis are considered to be positive; those to the left to be negative. Ordinates above the x axis are considered to be positive; those below to be negative.

The point P_1 in the above figure is designated either as $P_1(3, 4)$, or simply as $(3, 4)$; P_2 is designated as $P_2(-2, 3)$, or as $(-2, 3)$; P_3 is designated as $P_3(-3, -2)$, or as $(-3, -2)$; and P_4 is designated as $P_4(3, -1)$, or as $(3, -1)$.

When either the abscissa or the ordinate of a point is zero, the point is on one of the axes; for example, $(0, 3)$ is a point on the y axis, and $(4\frac{1}{2}, 0)$ is a point on the x axis. The origin O is the point $(0, 0)$.

Exercises. Plotting Points and Lines

Locate the position of each of the following points:

1. (2, 7). **5.** (− 4, 0). **9.** (3, 5). **13.** (2, 5).

2. (0, 6). **6.** (0, − 4). **10.** (3, − 5). **14.** (5, 2).

3. (6, 0). **7.** (4, 0). **11.** (− 3, 5). **15.** (5, − 2).

4. (6, 6). **8.** (− 4, − 4). **12.** (− 3, − 5). **16.** (− 5, − 2).

17. Plot the point having the abscissa 4 and the ordinate − 5; the abscissa − 5 and the ordinate 4.

18. Where is the point (0, 3)? the point (3, 0)? the point (0, − 7)? the point (− 7, 0)?

19. What is the distance between the points (3, 1) and (8, 1)? the points (− 3, 1) and (8, 1)? the points (2, 7) and $(2, -4\frac{1}{2})$?

20. How far from the origin is the point (3, 4)? the point (6, 8)? the point (4, − 3)? the point (− 6, 9)?

In Exs. 21–25 locate the position of all points

21. Whose ordinates are 3.

22. Whose abscissas are − 4.

23. Whose ordinates are equal to their respective abscissas.

24. Whose abscissas are numerically equal to their respective ordinates but have opposite signs.

25. Whose distance from the origin is always 5.

26. In making a survey the stations 0, 1, 2, · · · were taken 100 ft. apart in a straight line, and the elevations in feet above an established level were determined as shown in this table:

Station	0	1	1 + 50	2	2 + 25	3	4	5
Elevation	101.3	99.7	105.5	107.6	102.0	96.4	98.3	92.2

By plotting the elevations as ordinates, locate points for the eight observations, and then connect the points so as to show a profile of the section surveyed.

Function. In ordinary life we frequently speak of one quantity as depending upon one or more other quantities. For example, the time that it takes an athlete to run 100 yd. depends upon the speed at which he can run. Similarly, the load that a person can carry depends both upon his strength and the length of time that he has to carry it.

In mathematics, when one quantity depends upon another for its value, we say that the first is a *function* of the second.

For example, the expression $2x^2 + 5$ depends for its value upon the value of x, and hence is a function of x. Similarly, xy depends for its value upon both x and y, and is therefore a function of both x and y.

As we proceed, it will be found that a function of a variable often is typical of a whole class of expressions.

For example, the formula for finding the volume of a right circular cylinder is $V = \pi r^2 h$, where r is the radius of the base and h is the height. The volume V is therefore a function of both r and h.

Furthermore, this formula covers a large number of cases; for example, the cylinders may be long, short, wide, or narrow. The formula $V = \pi r^2 h$, however, is true for any one of these cases, provided only that the cylinder considered is of the type classified as a right circular cylinder.

In the formula $A = \pi r^2$, which gives the area of a circle, A is expressed as a function of r. We also see that as A varies in value r also varies; that is, r is a function of A, but its value is not explicitly stated unless we write the formula as $r = \sqrt{\frac{A}{\pi}}$. In the formula $A = \pi r^2$, A is said to be an *explicit function* of r, and r is said to be an *implicit function* of A.

Functional Notation. Instead of writing "$2x^2 + 5$ is a function of x" we may use the function symbol $f(\ )$, and write

$$f(x) = 2x^2 + 5.$$

The symbol $f(x)$ is read "function of x" or more simply "f of x." We may also use $F(x)$, $g(x)$, and so on, to indicate functions.

In finding the value of this function for various values of x, we may write

$$f(2) = 2 \cdot 2^2 + 5 = 8 + 5 = 13,$$

or $$f(3) = 2 \cdot 3^2 + 5 = 18 + 5 = 23, \text{ and so on.}$$

Constants. A quantity which does not change its value, regardless of the conditions of the problem into which it enters, is called a *constant.*

If a constant admits of no change whatever, it is called an *absolute constant.*

For example, 2, $-\frac{3}{4}$, $\sqrt{7}$, and 8^2 are absolute constants.

A quantity which may have various values assigned to it, but which maintains the same value throughout the same problem or discussion, is called an *arbitrary constant.*

In the equation $ax = 4$, a may have an indefinite number of values, for example, $\frac{1}{4}$, 1, 2, and 8, the corresponding values of x being 16, 4, 2, and $\frac{1}{2}$. If, however, we agree that a shall have some special value in a discussion (for example, 8), we arbitrarily make a a constant during the discussion and we therefore speak of it as an arbitrary constant.

Variables. A quantity which may have different values within certain limits set by the conditions of a problem or by the type of problem in which it occurs is called a *variable.*

For example, in the formula $A = \pi r^2$, which represents the area of a circle, π is a constant with the approximate value $3\frac{1}{7}$, but r and A are variables. The formula shows that these two variables are related to each other in such a way that $A = \pi r^2$ or $r = \sqrt{\frac{A}{\pi}}$.

Similarly, in the equation $y^2 = 5x$, x and y may have an indefinite number of pairs of values. If we limit the discussion to cases in which y must be real, then x is limited to positive values; if we limit the discussion to cases in which y must always be rational, then x must always be five times some square number. We may place other limitations upon the variables x and y if we so desire.

In the formula $C = 2\pi r$ we ordinarily think of C, the circumference, as depending upon r for its value, and we therefore call C the *dependent variable.* Under these conditions, r is considered as being able to have various values without regard to C, and is therefore called the *independent variable.*

When r is expressed as an explicit function of C, that is, as $r = \frac{C}{2\pi}$, r is regarded as being the dependent variable, and C consequently becomes the independent variable.

Exercises. Variables and Functions

Make a list of all the variables and all the constants that appear in the following formulas:

1. $A = lw$.
2. $A = bh$.
3. $s = \frac{1}{2}at$.
4. $A = s^2$.
5. $C = 2\pi r$.
6. $A = \pi r^2$.
7. $A = \frac{1}{2}\pi d^2$.
8. $V = lwh$.
9. $V = \pi r^2 h$.

10. In each of Exs. 1–9 state the quantity or quantities of which the left-hand member of each formula is a function.

Find the value of each of the following functions for the specified values of the variable:

11. $2x - 7$; $f(2)$, $f(-5)$.
12. $x^2 - 4$; $f(0)$, $f(4)$.
13. $2x^2 + 7$; $f(1)$, $f(-1)$.
14. $x^2 + 5x + 6$; $f(1)$, $f(-2)$.
15. $2x^2 + 3x - 7$; $f(-2)$, $f(\frac{1}{2})$.
16. $2x^3 + x^2 - 10$; $f(2)$, $f(-1)$.

17. In the formula $A = \frac{1}{2}(a + b)h$, which gives the area of a trapezoid, of what is A a function? What variables and what constants appear in the formula?

18. Write the formula $A = \frac{1}{2}bh$ in a form in which b is expressed as an explicit function of A and h.

19. In the formula $V = \pi r^2 h$ what is the value of V when $h = 5$, $r = 7$? when $h = 10$ and $r = 14$? when $h = 20$ and $r = 14$?

20. What happens to the volume of a circular cylinder when the radius is doubled? when the height is doubled? when both the height and the radius are doubled?

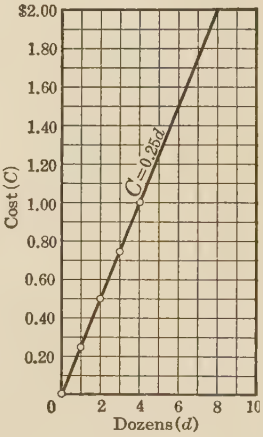
21. What happens to the volume of a rectangular solid if its length is doubled? if the width is doubled? if all three dimensions are doubled?

22. By finding the values of $f(x)$ for the values $0, 1, \cdots, 5$ of x show that $f(x)$ is the functions $f(x) = x$, $f(x) = x^2$, and $f(x) = 2^x$ do not increase at the same rate, and state the reason for the difference.

Graph of a Formula. As we have just seen, a formula is an expression of a functional relationship. For example, let us take the simple case of the formula

$$C = 0.25\,d,$$

which represents the cost C of a number of dozen d of peaches at 25¢ per dozen.



In this case C is obviously an explicit function of d; or, as we say in everyday language, the cost of purchasing peaches in dozen lots depends upon the number of dozens purchased.

In studying the functional relationships involved in formulas it is very convenient to draw a graph, which may be regarded as a sort of picture of the particular relationship represented.

In the graphs which we have drawn thus far we have generally been given a table of statistics from which to plot the corresponding points. In order to obtain such a table in this case we may assign convenient values to d and then compute the corresponding values of C, as shown in the following table:

If $d=$	0	1	2	3	4	···
$C=$	0	0.25	0.50	0.75	1.00	···

Plotting the values of d on the x axis and the values of C on the y axis, we may then draw the graph shown above. In this case the graph turns out to be a straight line. In fact, as we shall learn later, in any functional relationship of the form $f(x) = kx$, where k is a constant like 0.25 in the above formula, the graph is always a straight line.

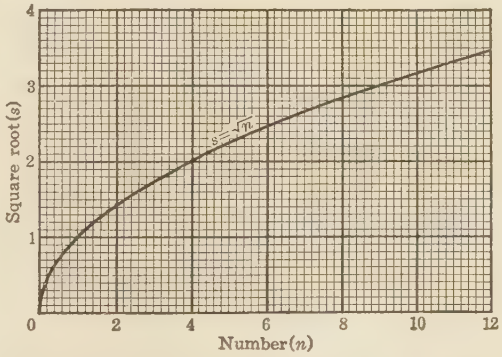
Considerable time can often be saved by examining the nature of the functional relationship before starting to draw the graph.

Interpreting a Graph. From a graph like the one on page 52 we can generally draw one or more conclusions of importance. For example, since the graph is a straight line, we see that the cost of the peaches increases uniformly as the number of dozens purchased increases; that is, if the number of dozens is doubled, the cost is doubled; if the number is tripled, the cost is tripled, and so on.

The graph can also be used as a means of determining quickly the cost of any number of dozens represented.

For example, by following along the ordinate at the point for which $d = 5$, we see that it cuts the graph at the point for which $C = 1.25$; that is, 5 doz. peaches cost \$1.25. Similarly, 7 doz. peaches cost \$1.75; $3\frac{1}{2}$ doz. peaches cost 88¢, and so on.

The importance of a graph as a means of saving computation is today being widely recognized; in fact, one of the most significant advances of modern mathematics is the increased attention paid to graphic representation. From the accompanying graph, for example, it is possible to read to a degree of accuracy sufficient for many purposes the square roots of numbers from 0 to 12.



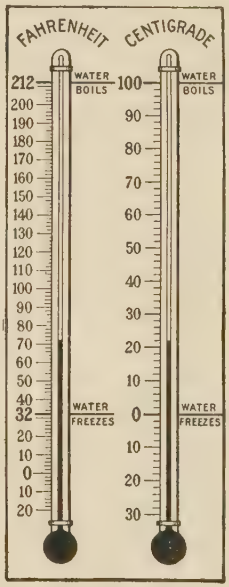
In drawing a graph like this, which is a graph of the formula

$$s = \sqrt{n},$$

it will be found convenient to use the table of roots given on page 302, taking the values of s to the nearest 0.01.

In the graph on page 52 each square on the x axis represents 1 doz., while each square on the y axis represents 10¢. In the above graph each of the centimeter squares represents 2 on the x axis and 1 on the y axis. This need for choosing convenient scales will be found in all kinds of graphic representation. The graph should be clearly marked so that the scale used in drawing it will be evident.

Temperature Graph. Another illustration of the convenience of a graph as a means of saving computation is found in the two scales for measuring temperature in use in this country. The first is the common Fahrenheit scale used in our house and outdoor thermometers, and the second is the centigrade scale used in scientific work.



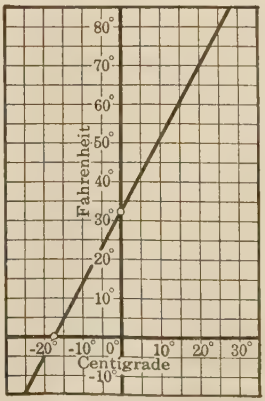
The differences between these scales will be apparent from the illustration at the right. On the centigrade scale there are 100 divisions between the freezing point 0° and the boiling point 100°, while on the Fahrenheit scale there are 180 such divisions, since the freezing point is 32° and the boiling point is 212°.

A temperature on the Fahrenheit scale is therefore 1.8 times the reading on the centigrade scale plus 32°, which we may express by means of the formula

$$F = 1.8\,C + 32.$$

Although the above formula may be used for converting any thermometer reading from one scale to the other, we shall find a graph of the formula more convenient.

We may draw the graph by first locating the point for which $C = 0°$ and $F = 32°$ and then the one for which $F = 0°$ and $C = -17.8°$. A third point, such as the one for which $C = 10°$ and $F = 50°$, may be located as a check.



From the graph it is possible to read at once the temperature on one scale which corresponds to any given temperature on the other.

Thus, 25° C. corresponds to 77° F., 5° C. corresponds to 41° F., and 10° F. corresponds to −12° C.

Exercises. Graphs of Formulas

1. The cost of sending a package by parcel post to the third zone is given by the formula $C = 2w + 6$, where C represents the cost in cents and w represents the weight in pounds. Draw a graph of this formula and show how it can be used.

2. A printer finds that the cost C, in dollars, of printing n hundreds of a certain kind of pamphlet is given by the formula $C = 2n + 5$. Draw a graph of this formula and find C when $n = 5$; when $n = 15$; when $n = 22$.

3. Taking the value of π as $3\frac{1}{7}$, draw a graph of the formula $C = 2\pi r$. What does the graph show with regard to the rate at which the circumference of a circle increases as the radius increases? How does the graph show this?

4. On the same axes and to the same scale used in Ex. 3 draw the graphs of the formulas $A = \pi r^2$ and $V = \frac{4}{3}\pi r^3$, which give respectively the area of a circle and the volume of a sphere, each with the radius r. Write a statement comparing the ways in which C, A, and V increase as r increases.

5. On squared paper draw a Fahrenheit-centigrade graph to a scale two and a half times that of the one on page 54.

6. From the graph of Ex. 5 find the Fahrenheit readings corresponding to 23° C., 15° C., 8° C., and − 4° C.

7. From the graph of Ex. 5 find the centigrade readings corresponding to 62° F., − 10° F., 18° F., and 80° F.

8. The most healthful temperature for a living room is 68° F. What is the corresponding centigrade reading?

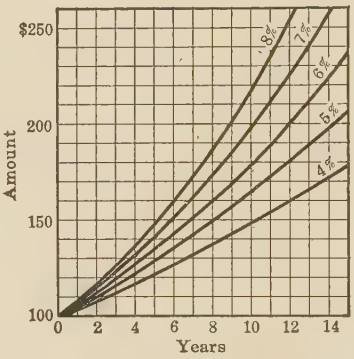
9. Draw a graph that can be used for converting centimeters to inches (1 in. = 2.54 cm.), and from it find the length in inches of 10 cm.; of 2.5 cm.; of 12 cm.

10. Draw a graph that can be used for converting pounds to kilograms (1 kg. = 2.2 lb.), and from it find the weight in pounds equivalent to 10 kg.; to 18 kg.

11. Draw a graph of the formula $c = \sqrt[3]{n}$, which can be used in finding the cube root c of a number n, for integral values of n from 0 to 12. From the graph find the value of $\sqrt[3]{2.5}$; of $\sqrt[3]{4.7}$; of $\sqrt[3]{10.9}$.

In drawing this graph refer to the table on page 302.

12. The accompanying figure shows how an investment of \$100 increases itself at different rates when the interest as it comes due is reinvested at the same rate. From the graphs find the approximate number of years in which the investment will double itself at 5%; at 6%; at 7%; at 8%.



The figure shows the graphs of the formula $A = P(1 + i)^n$ when $P = 100$ and i has in turn the values 0.04, 0.05, $\cdots$, 0.08. That is, the graph marked 4% is the graph of $A = 100 \times 1.04^n$ for values of n from 0 to 15.

13. From the graph of Ex. 12 determine the approximate amount of interest paid in 10 yr. on an investment of \$100 at each of the five rates of interest.

14. A factory finds that the number w of workmen needed to turn out a certain number m of machines in a given number d of days is given by the formula $w = \frac{6m}{d}$. On the same axes draw a graph for each of the values 1, 2, 3, 4, and 5 of d.

15. A company computes the shipping charges C on small parcels by the formula $C = \frac{1}{100} w(p - 30) + 30$, where w is the weight in pounds and where p is the rate per 100 lb. which applies only to parcels weighing 100 lb. and over. To seven different cities the rates p are 40¢, 50¢, 60¢, 70¢, 80¢, 90¢, and \$1 respectively. On the same axes draw a graph for each of these values of p, and write a statement that will show how the graph can be used.

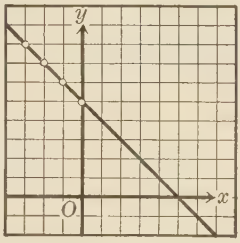
Graph of a Linear Equation. In Chapter II we learned that

$$x + y = 5$$

is a linear equation in both x and y. By assigning convenient values to x in the given equation, we may compute the corresponding values of y as shown in the following table of values:

If $x =$	$\cdots$	-3	-2	-1	0	1	2	3	4	5	6	$\cdots$
$y =$	$\cdots$	8	7	6	5	4	3	2	1	0	-1	$\cdots$

After plotting the first few points, as shown in the figure below, it becomes apparent that the graph of the equation will be a straight line. We can now see why the word "linear," which means "in a straight line," is applied to this type of equation.



In higher mathematics it is proved that the graph of a linear equation in x and y is always a straight line, but we do not need to prove it here.

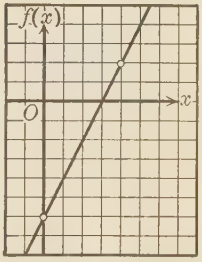
Since there is an unlimited number of pairs of values that satisfy the equation, no one of them is a solution. Such an equation is therefore said to be *indeterminate*.

As an example of a determinate linear equation we may take

$$2x - 6 = 0.$$

Since $2x - 6$ is a function of x, we have

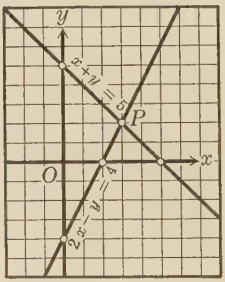
$$f(x) = 2x - 6.$$



Now when $x = 0$, $f(x) = -6$, and when $x = 4$, $f(x) = 2$. By plotting these two points and drawing a line through them, we can construct the graph here shown.

The graph cuts the x axis at the point for which $x = 3$. Since this is a point for which $f(x) = 0$, we thus find graphically that 3 is a root of the equation $2x - 6 = 0$, as can be easily verified by substituting 3 for x in the equation.

Graphs of Two Simultaneous Linear Equations. If on the same axes used for the graph of the equation $x + y = 5$ on page 57 we now draw the graph of the equation $2x - y = 4$, we shall have the figure shown below. The two graphs intersect at the point P for which $x = 3$ and $y = 2$. Since this point of intersection is common to both graphs, the coordinates of this point satisfy both equations. We thus see that the equations



$$x + y = 5 \qquad (1)$$

and

$$2x - y = 4 \qquad (2)$$

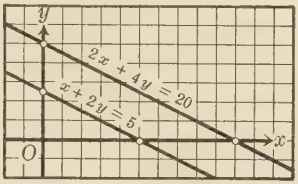
are simultaneous equations (page 23) which are satisfied by the common pair of values $x = 3$ and $y = 2$.

The easiest way of constructing the graph of a linear equation is to plot two points and then draw a line through them. In general, we shall find it most convenient first to let $x = 0$ and compute the value of y, thus locating a point on the y axis; we may then let $y = 0$ and compute the value of x, thus locating a point on the x axis. A third point, however, should be located as a check on the accuracy of the work, as suggested on page 54.

By examining the above figure it is evident that there is an unlimited number of points which lie on each graph, but there is only one point — the point (3, 2) — which lies on both graphs. Similarly, there is an unlimited number of pairs of values which satisfy each equation, but there is only one pair — the values $x = 3$ and $y = 2$ — which satisfy both equations.

If the values of x and y in two simultaneous equations were always small integers, we could solve such a pair of equations by simply drawing their graphs and determining the coordinates of their intersection. Since this is not the case, we frequently have to resort to the algebraic methods given in Chapter II. A graph, however, may serve as a convenient check upon an algebraic solution, and it often enables us to see some of the properties of the equations that we might not see in examining the equations from a purely algebraic standpoint.

Inconsistent Equations. The graphs of the equations on page 58 intersect at a common point, and the coordinates of this point satisfy both equations. Ordinarily, any two straight lines in a plane will intersect if they are extended far enough. If, however, we construct the graphs of the equations



$$x + 2y = 5 \qquad (1)$$

and

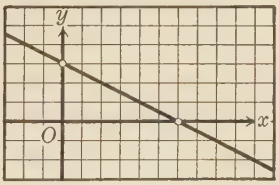
$$2x + 4y = 20, \qquad (2)$$

we shall find, as here shown, that the two graphs are parallel. They therefore cannot intersect, and hence there is no point whose coordinates satisfy this pair of equations. In other words, the equations are not simultaneous and therefore have no common solution.

If the graphs of two linear equations are parallel, the equations are said to be *inconsistent.*

This fact is also evident from the equations themselves. If in the second equation we divide by 2, we have $x + 2y = 10$, a statement which obviously cannot be true under the same conditions as the first equation, which states that $x + 2y = 5$. On this account such equations are sometimes referred to as being *contradictory equations.*

Equivalent Equations. If we draw the graphs of the pair of linear equations



$$x + 2y = 6 \qquad (1)$$

and

$$2x + 4y = 12, \qquad (2)$$

we shall find, as here shown, that one graph coincides with the other. Such equations are said to be *equivalent.*

Since the graphs coincide, they have an indefinite number of pairs of values of x and y in common, but there is no one pair that we can designate as the solution.

This fact is also evident from the equations themselves, because the second equation is the same as the first except for the factor 2; that is, if we divide by 2, we have $x + 2y = 6$. There is therefore no use in trying to solve a pair of equations whose graphs coincide.

Exercises. Linear Equations

Draw the graph of each of the following equations:

1. $x = y$.
2. $x = -y$.
3. $y = -x$.
4. $y = 2x$.
5. $y = 3x$.
6. $y = x + 2$.
7. $y = x - 2$.
8. $y = 2x + 1$.
9. $y = 2x - 1$.
10. $y = -x + 1$.
11. $y = 2x + 5$.
12. $x = 2y + 5$.
13. $5 = x + 2y$.
14. $y = \frac{1}{2}x + 1$.
15. $2y = 3x - 1$.

16. By means of a graph solve the equation $3x + 9 = 0$.

As in the second case on page 57, we may let $f(x) = 3x + 9$ and then draw the graph of this function. The abscissa of the point of intersection of the graph with the x axis is the value of x for which $f(x) = 0$.

Solve each of the following equations graphically:

17. $3x = 0$.
18. $y = 0$.
19. $2x - 4 = 0$.
20. $3x - 2 = 0$.
21. $2x = 7$.
22. $5x - 2 = 0$.

Solve the following pairs of equations graphically, or show that they are either inconsistent or equivalent:

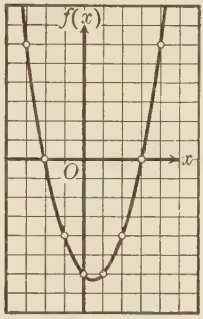
23. $x + y = 7$
    $x - y = 1$
24. $x + 2y = 8$
    $3x - y = 3$
25. $2x - 3y = 4$
    $4x - 6y = 5$
26. $3x - 4y = 2$
    $x - 1\frac{1}{3}y = \frac{2}{3}$
27. $2x - 7y = -6$
    $8x + 3y = 38$
28. $8x + y = 10$
    $3x - 2y = -1$
29. $2x + y = 17$
    $2y + x = 13$
30. $x - 7y = 26$
    $7x - y = 38$
31. $\frac{1}{2}x - 2y = 12$
    $\frac{3}{4}x + 3y = -30$
32. $2x - 7y = 4$
    $x - 3.5y = 2$
33. $8x + 2y = 9$
    $4x + y = 6$
34. $x + y = 0$
    $19x + 7y = -132$
35. $2x + 4y = -15$
    $3x + 6y = 13$
36. $7x + 3y = 9$
    $2x + \frac{6}{7}y = 2\frac{4}{7}$

Graph of a Quadratic Equation. We may also use graphs in the solution of a quadratic equation; for example, in the case of

$$x^2 - x - 6 = 0,$$

we see that $x^2 - x - 6$ is a function of x, and hence we may write

$$f(x) = x^2 - x - 6.$$



By assigning convenient values to x we may then construct the following table of values:

$x =$	-3	-2	-1	0	1	2	3	4
$f(x) =$	6	0	-4	-6	-6	-4	0	6

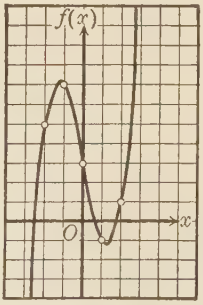
Drawing the graph, as shown above, we find that it cuts the x axis at the points for which $x = -2$ and $x = 3$; that is, since $f(x) = 0$ at these points, the roots of the equation are -2 and 3, as can be easily verified by substitution.

The graph of every quadratic of the form $ax^2 + bx + c = 0$ is a curve of the same general shape as the one shown above, which is called a *parabola*.

By the method of graphs we can find only the real roots of a quadratic equation. For example, the method will not enable us to find that the roots of the equation $x^2 + 9 = 0$ are $+3\sqrt{-1}$ and $-3\sqrt{-1}$.

Graph of a Cubic Equation. By drawing the graph of the cubic equation

$$x^3 - 5x + 3 = 0,$$



as shown in the figure at the right, we can find that approximate values of the three roots, which are all real in this case, are -2.5, 0.7, and 1.8.

That is, letting $f(x) = x^3 - 5x + 3$, we first make a table of values, as in the above case, then draw the graph, and find the abscissas of the three points for which $f(x) = 0$.

In this case we can find only approximate values for the roots, the accuracy of the results depending in a large part upon the size of the graph and the care with which it is drawn.

Graphs of Simultaneous Equations. Just as the graph of a linear equation in x and y is always a straight line, so the graph of a quadratic equation in x and y is always some form of one of the four types of smooth curves which are shown below:

Circle Ellipse Parabola Hyperbola

A special case of the hyperbola consists of two straight lines which intersect each other at the origin.

By the method of graphs we can therefore find the pairs of real values, or at least approximations to these values, that satisfy a pair of equations in which one is quadratic and the other linear or in which both are quadratic.

The figures below show four of the many different combinations of lines and curves that we may have in solving equations by this method. The first two figures illustrate typical cases of solving a linear and a quadratic equation, and the last two illustrate the solutions of two pairs of simultaneous quadratics:

In the case of a linear and a quadratic equation there are not always two points of intersection, and consequently two pairs of real values of x and y. For example, in the graphic solution of the pair of equations $3x^2 + 4y^2 = 12$ and $x + 2y = 4$ the line $x + 2y = 4$ will be found to be tangent to the ellipse $3x^2 + 4y^2 = 12$ at the point $(1, 1\frac{1}{2})$, and hence the values that satisfy the equations are $x = 1$ and $y = 1\frac{1}{2}$.

Similarly, in the case of two quadratics there are not necessarily four points of intersection; for example, in the case of the circle $x^2 + y^2 = 25$ and the parabola $y^2 = 2\frac{1}{4}x$ there are only two points of intersection, the coordinates of which are $x = 4$, $y = 3$ and $x = 4$, $y = -3$. There may, of course, be only one or even no points of intersection.

Exercises. Solution by Graphs

Solve each of the following equations graphically:

1. $x^2 - 2x - 3 = 0$.
2. $x^2 - 2x - 8 = 0$.
3. $x^2 - 4x - 5 = 0$.
4. $x^2 - 10x + 25 = 0$.
5. $2x^2 - 3x - 2 = 0$.
6. $6x^2 - 7x + 2 = 0$.
7. $x^3 - 4x^2 + x + 6 = 0$.
8. $x^3 - 2x^2 - 8x = 0$.
9. $x^3 + 2x^2 - x - 2 = 0$.
10. $x^3 + x^2 - x - 1 = 0$.
11. $2x^3 - x^2 - 12x + 6 = 0$.
12. $x^4 - 5x^2 + 4 = 0$.

Solve the following pairs of equations graphically:

13. $x^2 + y^2 = 17$
    $x + y = 3$
14. $2x^2 + y^2 = 33$
    $x - y = -3$
15. $3y^2 = x$
    $x^2 + 3y^2 = 12$
16. $xy = -8$
    $x^2 + y^2 = 20$

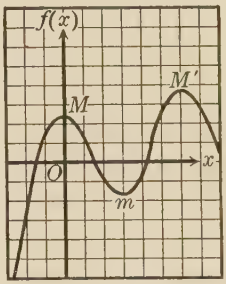
17. In a series of crop experiments it was desired to lay out a 5-acre field that should be 10 rd. longer than $2\frac{1}{2}$ times its width. Find the dimensions of the field.

18. A box containing exactly 1 pk., which is 537.6 cu. in., is made by cutting squares 4 in. on a side from the corners of a rectangular piece of tin $1\frac{1}{2}$ times as long as it is wide and then turning up the flaps. What were the dimensions of the original piece of tin?

19. In physics it is shown that the depth of flotation in water of a sphere with the radius r and the specific gravity s is one of the positive roots of the equation $z^3 - 3rz^2 + 4r^3s = 0$. Find the value of z when $r = 1$ and $s = 0.65$.

20. For liquids other than water the s in the equation in Ex. 19 is replaced by $\frac{s}{s_1}$, in which s is the specific gravity of the sphere and s_1 is the specific gravity of the liquid. Find the depth of flotation of a porcelain ball used as an indicator in storage batteries when $r = 1$, $s = 1.15$, and $s_1 = 1.275$.

Maximum and Minimum Values. A function $f(x)$ is said to have a *maximum value* when $x = a$ if $f(a)$ is greater than any other value of $f(x)$ in that vicinity.



For example, in the graph of this function, the maximum values are evidently at M and M', for which $x = 0$ and $x = 6$ respectively.

Similarly, $f(x)$ is said to have a *minimum value* when $x = a$ if $f(a)$ is less than any other value of $f(x)$ in that vicinity.

In the above case the minimum value is at m, for which $x = 3$. There may, of course, be more than one minimum value for the function.

Maximum and minimum values have many important applications in mathematics, physics, and mechanics.

Exercises. Maximum and Minimum Values

Find the approximate maximum or minimum values of each of the following functions of x by means of a graph:

1. $x^2 - 5x + 6$. **4.** $2x^2 + 3x + 1$. **7.** $-2x^2 + x + 10$.

2. $x^2 + 5x - 6$. **5.** $3x^2 - 3x - 2$. **8.** $5 + 8x - 8x^2$.

3. $10 + 7x - x^2$. **6.** $-3x^2 - 2x + 5$. **9.** $2x^2 - 3x - 4$.

10. In the formula $P = 2c - 0.01c^2$ determine the value of c which makes P a maximum.

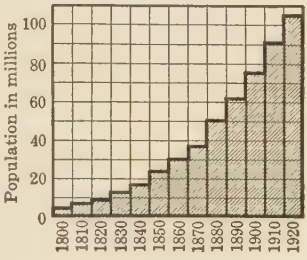
11. A rectangular garden is to be inclosed by 300 ft. of fencing. What dimensions will give the maximum area?

12. An open rectangular tank twice as long as it is wide is to be made from 54 sq. ft. of galvanized sheet iron. Making no allowance for waste, find the dimensions of the tank which will have the maximum capacity.

13. The parcel-post regulations require that the combined length and girth of a package shall not exceed 84 in. Find the dimensions and the volume of the largest rectangular package with square ends that can be shipped by parcel post.

Exercises. Test on Chapter III

1. From this graph state the probable population of the United States in 1885; in 1902; in 1915; in 1918; in 1925.



Population of the United States 1800–1920

2. State upon what each of the following depends: (1) the circumference of a circle; (2) the time it takes an airplane to go 100 mi.; (3) the time it takes a boy to run 5 mi.; (4) the cost of an overcoat; (5) the horse power of an automobile engine.

3. In recording the growth of a plant during successive weeks, which kind of graph should you use, a bar graph or a curve-line graph? Give reasons for your answer.

4. In recording the variations in rainfall over a period of several months, why is it better to use a bar graph instead of a curve-line graph?

5. An open rectangular box of maximum capacity is to be made from a rectangular sheet of tin 24 in. long and 16 in. wide by cutting squares from the corners and turning up the flaps. What will be the size of the squares?

6. A farmer has 5280 ft. of fencing with which he plans to fence in the largest possible rectangular pasture, using a stone wall as one side of the pasture. What will be the area and the dimensions of the pasture?

7. The formula for the volume V of a cube in terms of T, the area of the total surface, is $V = (\frac{1}{6} T)^{\frac{3}{2}}$. From this formula derive one for T in terms of V.

8. Write a formula for computing the wages of a factory employee who is paid at the rate of 75¢ per hour. Draw a graph of this formula, and from the graph find the amount of wages paid for 3 hr.; 5 hr.; $2\frac{1}{2}$ hr.; $3\frac{3}{4}$ hr.

Exercises. General Review

Perform the following indicated operations:

1. $(x^2-2xy+y^2)(x^2+2xy+y^2)$.
2. $3x^2-4x+7-(2x^2-7x-9)$.
3. $\dfrac{x^3+3x^2y+3xy^2+y^3}{x+y}$.
4. $(x^2-7y^3)(x^2+7y^3)$.
5. $(x^3+1)\div(x+1)$.
6. $\dfrac{x+y}{x-y}\cdot\dfrac{x^2-y^2}{2x+2y}$.

Solve the following pairs of equations by means of graphs and check by substituting the results in the original equations:

7. $5x+2y=17y$, $3x-y=2$
8. $2x+y=13$, $5x-2y=1$
9. $x^2+y=16$, $x^2-y=2$

Find the factors of the following expressions:

10. $9x^2-49y^2$.
11. $xy^2+xy-72x$.
12. $1-8y+16y^2$.
13. x^2+x-72.
14. $5m^2-80$.
15. $a^2-2a-24$.
16. $6a^2+7a+2$.
17. $x^2+46x+45$.
18. $x^2+37x+322$.

Solve each of the following equations:

19. $x^2+12x=133$.
20. $2y^2-15y=27$.
21. $\frac{1}{2}n^2+6\frac{3}{4}n=3\frac{1}{2}$.
22. $2x^2+3=7x$.
23. $2x(x-1)=1\frac{1}{2}$.
24. $x^2+4x-1=0$.
25. $2x^2+3x-6=0$.
26. $10y^2+3ay-a^2=0$.
27. $y^2-2ay=b^2-a^2$.
28. $x(x+5)=-6$.

29. Give an illustration of a case in which it would be helpful to draw two or more graphs to the same scale on the same axes.

30. If, when drawn to the same scale on the same axes, the graph of a linear equation fails to cut the graph of a quadratic equation, what does this show?

31. If the prices of different sizes of oranges depended upon their volumes, how much should be paid for oranges which measure 3 in. in diameter when oranges $2\frac{1}{2}$ in. in diameter are selling at 40¢ per dozen?

CHAPTER IV

LOGARITHMS

Computation by Logarithms. Arithmetic computations, particularly those involving multiplication, division, and the finding of roots, soon become tedious when it is necessary to use large numbers. One of the most important of the many devices invented for reducing the labor of computation is known as the *method of logarithms*, which depends upon the familiar laws of algebraic exponents.

Logarithms were invented by John Napier, a Scotchman, early in the seventeenth century. Napier's tables were later much improved by Henry Briggs, one of his contemporaries.

Logarithm Defined. It is evident that we may, if we wish, regard all positive numbers as powers of some particular positive number, which is called the *base*.

For example, if we use the base 2, we have $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, and so on; if we use the base 4, we have $16 = 4^2$, $64 = 4^3$, $256 = 4^4$, and so on.

Thus, if we let N represent any positive number and b the positive number taken as the base, we have

$$N = b^l.$$

The exponent l to which the base b is raised in order to equal the number N is called the *logarithm of N to the base b*, which is usually abbreviated in writing to "$\log_b N$." The above definition of a logarithm may therefore be expressed as

$$\log_b N = l.$$

As in the case of Chapter III, the work of this chapter may have been already covered by some students who have studied logarithms in elementary algebra. The work should therefore be assigned in accordance with the individual needs of the class.

Numbers as Powers of 10. Although any positive number may be taken as the base of a system of logarithms, in our work we shall use only the base 10; that is, any logarithm that we shall use will always be an exponent of 10.

For our purposes, the two forms of the definition on page 67 may be written as $N = 10^l$ and $\log_{10} N = l$.

It will therefore not be necessary to indicate the base each time; that is, log 3 will always mean $\log_{10} 3$.

Thus, since $100 = 10^2$,

we have $\log 100 = 2$.

In the same way, $1000 = 10^3$,

and hence $\log 1000 = 3$.

Similarly, from the table of roots on page 302 we find that

$$\sqrt{10} = 10^{\frac{1}{2}} = 3.162,$$

and hence $\log 3.162 = \frac{1}{2} = 0.5000$.

By making a table of a few powers of 10 we have the following relations:

$10^0 = 1$	$\log 1 = 0$	$10^{-1} = \frac{1}{10} = 0.1$	$\log 0.1 = -1$
$10^1 = 10$	$\log 10 = 1$	$10^{-2} = \frac{1}{100} = 0.01$	$\log 0.01 = -2$
$10^2 = 100$	$\log 100 = 2$	$10^{\frac{1}{2}} = 3.162$	$\log 3.162 = 0.5000$
$10^3 = 1000$	$\log 1000 = 3$	$10^{\frac{1}{3}} = 2.154$	$\log 2.154 = 0.3333$

We thus see that any positive number can be expressed as a power of 10, either exactly or approximately.

In working with logarithms, it should be remembered that

Logarithms are exponents, and hence all the laws of exponents apply to logarithms.

For example, in the case of $\log \sqrt{10} = 3.162$, above, we see that

$31.62 = 10 \times 3.162$;

that is, $31.62 = 10^1 \times 10^{\frac{1}{2}} = 10^{\frac{3}{2}}$,

and hence $\log 31.62 = \frac{3}{2} = 1.5000$.

Similarly, $316.2 = 100 \times 3.162$;

that is, $316.2 = 10^2 \times 10^{\frac{1}{2}} = 10^{\frac{5}{2}}$,

and hence $\log 316.2 = \frac{5}{2} = 2.5000$.

Characteristic and Mantissa. From the previous discussion we see that a logarithm may consist of

1. An integer, as in $\log 1000 = 3$;
2. A decimal, as in $\log 3.162 = 0.5000$;
3. An integer plus a decimal, as in $\log 316.2 = 2.5000$.

The part of the logarithm at the left of the decimal point (the integral part) is called the *characteristic.*

For example, in the case of $\log 316.2 = 2.5000$ the characteristic is 2; in the case of $\log 3.162 = 0.5000$ the characteristic is 0, and so on.

The decimal part of the logarithm is called the *mantissa.*

Thus, in the case of $\log 316.2 = 2.5000$ the mantissa is .5000, and in the case of $\log 1000 = 3$, which may be written $\log 1000 = 3.0000$, the four figures of the mantissa are all zeros.

The customary form of writing $\frac{1}{2}$ as a four-place decimal is 0.5000, but in referring to a mantissa we shall use the form .5000 in order to avoid possible confusion with a zero characteristic.

Tables of Logarithms. In practical work the logarithm of a number is found from a table like the one on pages 314 and 315. In this table each logarithm is given to the nearest 0.0001, and hence the table is called a *four-place table.*

Later on in this book, particularly in connection with the investment problems in Chapter VII, it will be necessary to use the six-place table of logarithms which is given on pages 316–333. This table gives more accurate results than can be obtained from the four-place table.

In order not to confuse the student, the work in the present chapter is limited strictly to the four-place table, which gives results that are sufficiently accurate for ordinary work. Instruction in the use of the six-place table, which is described on page 313, may well be postponed until the table is needed in the special work given later.

It should always be remembered that logarithms are only approximate, and hence the results of a computation in which logarithms are used will be only approximate. A four-place table will not give results that are accurate to more than four figures, and even the fourth figure is likely to be approximate.

Finding the Characteristic. From the table of powers of 10 given on page 68 we see that the logarithm of any number between 1 and 10 is 0 plus some mantissa; the logarithm of any number between 10 and 100 is 1 plus some mantissa; and so on.

We also see that any number between 0 and 1 can be expressed as a negative power of 10; that is,

The logarithm of a proper fraction is negative.

For example, $0.1 = \frac{1}{10} = 10^{-1}$, $0.01 = \frac{1}{100} = 10^{-2}$, and so on.

We therefore have the following rules for finding the characteristic of a logarithm:

1. *The characteristic of the logarithm of a number greater than 1 is positive, and is one less than the number of integral places in the number.*

For example, log 275.36 = 2 + some mantissa.

2. *The characteristic of the logarithm of a number between 0 and 1 is negative, and is numerically one more than the number of zeros immediately following the decimal point.*

For example, log 0.002 = − 3 + some mantissa.

We can therefore tell the characteristic of the logarithm of a number by inspection. To find the mantissa, however, we consult the table, which will be explained later.

Writing a Negative Characteristic. Since 0.05 is a proper fraction, we have

$$\log 0.05 = -2 + \text{some mantissa},$$

which, as we shall presently find, is .6990; that is,

$$\log 0.05 = -2 + .6990,$$

or

$$\log 0.05 = 0.6990 - 2.$$

The usual way of writing such a logarithm is

$$\log 0.05 = \bar{2}.6990,$$

which indicates that the characteristic 2 is negative while the mantissa .6990 is positive.

Exercises. Meaning of Logarithms

1. What is the meaning of the expression $10^2 = 100$? of the expression $\log 100 = 2$?

2. Taking the value of $10^{\frac{3}{4}}$, or $10^{0.75}$, as 5.624, to the nearest 0.001, to what power of 10 is 5.624 equal? What is the logarithm of 5.624?

Write each of the following expressions in the form $\log 100 = 2$:

3. $10^1 = 10$.
4. $10^3 = 1000$.
5. $10^{1.6} = 39.81$.
6. $10^{2.9} = 794.3$.
7. $10^{0.50} = 3.162$.
8. $10^{1.45} = 28.19$.

Write each of the following expressions in the form $10^2 = 100$:

9. $\log 4 = 0.6021$.
10. $\log 30 = 1.4771$.
11. $\log 1.6 = 0.2041$.
12. $\log 814 = 2.9106$.
13. $\log 36 = 1.5563$.
14. $\log 360 = 2.5563$.

State the characteristic and the mantissa of the logarithm in

15. $\log 70 = 1.8451$.
16. $\log 0.7 = \bar{1}.8451$.
17. $\log 700 = 2.8451$.
18. $\log 4.2 = 0.6232$.
19. $\log 0.042 = \bar{2}.6232$.
20. $\log 4200 = 3.6232$.
21. $\log 3.76 = 0.5752$.
22. $\log 736 = 2.8669$.

State the characteristic of the logarithm of

23. 3.
24. 30.
25. 0.03.
26. 0.30.
27. 80.
28. 0.8.
29. 0.008.
30. 0.080.
31. 125.
32. 1.25.
33. 0.125.
34. 12.50.
35. 0.65.
36. 650.
37. 6500.
38. 60.50.
39. 0.013.
40. 1310.
41. 13.15.
42. 267.8.

State the number of integral places in the number whose logarithm has the characteristic

43. 2.
44. 1.
45. 0.
46. 3.
47. 4.

State the number of zeros immediately following the decimal point in the number whose logarithm has the characteristic

48. -2.
49. -1.
50. -5.
51. -6.
52. -3.

Finding Logarithms from the Table. The use of the four-place table of logarithms given on pages 314 and 315 will be evident from the following illustrative examples:

1. Find the logarithm of 1.34.

In the column headed **N** on page 314 we first look for **1.3**. Then following along horizontal line **1.3** to the column headed **4**, we find .1271. Since the characteristic, which is always 0 for any number with one integral place, is omitted in the table, we thus find that

$$\log 1.34 = 0.1271.$$

2. Find the logarithm of 13.47.

As in Ex. 1, we find that $\log 1.34 = 0.1271$. Now $13.4 = 10 \times 1.34$, and hence, as on page 68, we see that

$$\log 13.4 = 1.1271.$$

By following along horizontal line **1.3** to the columns at the extreme right, which are called *difference columns*, we find 23 in the column headed **7** (the fourth figure of 13.47). This means that log 13.47 is .0023 greater than log 13.4; that is,

$$\log 13.47 = 1.1271 + .0023 = 1.1294.$$

3. Find the logarithm of 0.06264.

In this case we see from the rule on page 70 that the characteristic of log 0.06264 is -2. Instead of using the difference columns as in Ex. 2, we may use the following method, which is known as *ordinary interpolation*. In this method we first find that

$$\log 0.0627 = \bar{2}.7973$$

and that

$$\log 0.0626 = \bar{2}.7966$$

Subtracting, the difference is 0.0007

Now 0.06264 lies 0.4 of the way from 0.0626 to 0.0627, and hence we may take 0.4 of .0007, or .0003, as the difference to be added to log 0.0626. That is, we thus find that

$$\log 0.06264 = \bar{2}.7966 + .0003 = \bar{2}.7969.$$

Although ordinary interpolation sometimes gives results that are more nearly accurate in the fourth decimal place, the difference columns are sufficiently accurate for most work. Since it is generally easier to use the difference columns, the student should feel free to do so. He should, however, be familiar with both methods of interpolation.

Finding Antilogarithms from the Table. We have now learned how to use the table to find the logarithm of a number. It is equally important to be able to find the number which corresponds to a given logarithm. This number is called the *antilogarithm*, which, in writing, is usually abbreviated to "antilog."

Thus, since log 43.52 = 1.6387, we know that antilog 1.6387 = 43.52.

The method of finding an antilogarithm is simply the inverse of that of finding a logarithm, as shown below.

1. Find the antilogarithm of 0.6075.

Looking through the table on pages 314 and 315, we find that .6075 is in horizontal line **4.0** and in column **5**; that is,

$$\text{antilog } 0.6075 = 4.05.$$

2. Find the antilogarithm of 1.6871.

Disregarding the characteristic at first, we look in the table for the mantissa .6871. Although we do not find it, we find that the next smaller mantissa listed is .6866, which lies in line **4.8** and column **6**; that is, .6866 is the mantissa of log 4.86, of log 48.6, of log 0.0486, and so on.

Since .6866 is .0005 smaller than .6871, we follow along line **4.8** to the figure 5, which is in the difference column **6**. We thus see that the fourth figure of the antilogarithm is 6.

Since the characteristic of the given logarithm is 1, we see that there are two integral places in the antilogarithm (see page 70); that is,

$$\text{antilog } 1.6871 = 48.66.$$

3. Find the antilogarithm of $\bar{2}.9501$.

Looking in the table for the mantissa .9501, as in Ex. 2, we find that .9499, the next smaller mantissa listed, is the mantissa of log 8.91. Since .9499 is .0002 smaller than .9501, we look for 2 in the difference columns and find that it is given on line **8.9** under both column 4 and column 5. We must therefore use ordinary interpolation (see Ex. 3, page 72).

Since the total difference between the mantissas of log 8.92 and log 8.91 is .0005, and .0002 is $\frac{2}{5}$, or .4, of this total difference, we see that .0002 corresponds to a fourth figure 4 (not 5) in the antilogarithm.

Since the characteristic of the given logarithm is -2, there will be one zero immediately following the decimal point in the antilogarithm (see page 70); that is, we thus find that

$$\text{antilog } \bar{2}.9501 = 0.08914.$$

Exercises. Logarithms and Antilogarithms

Find the logarithm of each of the following numbers:

1. 43.	**8.** 600.	**15.** 5.94.	**22.** 7200.
2. 69.	**9.** 602.	**16.** 0.549.	**23.** 5175.
3. 230.	**10.** 2061.	**17.** 0.0094.	**24.** 31,950.
4. 234.	**11.** 10.01.	**18.** 0.00008.	**25.** 0.1958.
5. 234.5.	**12.** 2.666.	**19.** 0.01056.	**26.** 99,990.
6. 17.98.	**13.** 2700.	**20.** 0.1573.	**27.** 0.09999.
7. 24.67.	**14.** 86,510.	**21.** 0.01010.	**28.** 0.001111.

Find the antilogarithms of the following logarithms:

29. 0.8432.	**35.** 0.7600.	**41.** 0.7330.	**47.** 0.8304.
30. 1.6425.	**36.** $\bar{1}$.9068.	**42.** 2.9570.	**48.** $\bar{1}$.3856.
31. $\bar{2}$.1987.	**37.** 2.6116.	**43.** $\bar{1}$.9010.	**49.** 1.6420.
32. 3.6884.	**38.** $\bar{3}$.2933.	**44.** 3.8246.	**50.** $\bar{2}$.8743.
33. $\bar{1}$.7672.	**39.** 0.5509.	**45.** $\bar{2}$.6834.	**51.** 2.9417.
34. 3.0000.	**40.** 1.0075.	**46.** 2.8598.	**52.** 0.6696.

53. Find log 0.00275 and divide it by 2; by 3; by 4; by 5.

From the table we have log 0.00275 = $\bar{3}$.4393 = 0.4393 − 3. This cannot be divided by 2 in the usual way because the division of − 3 by 2 gives a remainder. By writing log 0.00275 as 1.4393 − 4, however, we can divide exactly by 2, giving the mantissa of the result correct to the nearest 0.0001; that is, the result is 0.7197 − 2, or $\bar{2}$.7197.

Instead of the form 1.4393 − 4, we might use 7.4393 − 10 or 17.4393 − 20, and the result in each case would equal $\bar{2}$.7197. That is, in all cases where it is necessary to divide a logarithm having a negative characteristic, we simply select the part to be placed at the right in such a way that it is always exactly divisible.

54. Find log 0.5 and log 0.8357 and divide each logarithm by 3; by 4.

55. Find log 0.2956 and log 0.00275 and divide each logarithm by 4; by 5.

Multiplication with Logarithms. We shall now see how logarithms are used in shortening the labor of computation.

For example, if we take any two positive numbers a and b, and let l and l' be their respective logarithms, we have

$$a = 10^l, \quad \text{or} \quad \log a = l;$$

and

$$b = 10^{l'}, \quad \text{or} \quad \log b = l'.$$

Then

$$ab = 10^l \cdot 10^{l'} = 10^{l+l'},$$

and hence, by the definition of logarithm, we have

$$l + l' = \log ab,$$

or

$$\log ab = \log a + \log b.$$

This formula is easily proved for any base, and hence

The logarithm of the product of two numbers is the sum of the logarithms of the numbers.

Similarly, $\log abc = \log a + \log b + \log c$, and so on for other factors.

Although a negative number has no real logarithm, in a case like $-a \cdot b$ we know that $-a \cdot b = -ab$. Hence we can find the product of $a \cdot b$ by logarithms and prefix the minus sign.

In the equation $N = b^l$, the base b is always positive. Hence, if l is positive, N is positive; if the exponent is negative, as in $N = b^{-l}$, then $N = \frac{1}{b^l}$; if $l = 0$, then $N = b^0 = 1$. That is, there is no case in which N can be negative, which shows that a negative number cannot have a real logarithm.

Division with Logarithms. Taking the two numbers a and b, as above, we have

$$\frac{a}{b} = \frac{10^l}{10^{l'}} = 10^{l-l'};$$

whence

$$l - l' = \log \frac{a}{b},$$

or

$$\log \frac{a}{b} = \log a - \log b.$$

That is, in a case of division by means of logarithms,

The logarithm of a quotient is the difference between the logarithm of the dividend and the logarithm of the divisor.

Illustrative Examples. 1. Find the product of 2.75 and 37.6.

From the table, $\log 2.75 = 0.4393$

and $\log 37.6 = \underline{1.5752}$

Adding, we have 2.0145

Then $\text{antilog } 2.0145 = 103.4.$

Multiplying in the usual way, we have $2.75 \times 37.6 = 103.4$; that is, the two results are identical.

2. Find the quotient of $10.68 \div 13.02$.

From the table, we find that $\log 10.68 = 1.0286$ and $\log 13.02 = 1.1145$. In order to avoid a negative mantissa in the result, we proceed as follows:

$\log 10.68 = 2.0286 - 1$

$\log 13.02 = \underline{1.1145}$

Subtracting, we have $0.9141 - 1$

Then $\text{antilog } \bar{1}.9141 = 0.8206.$

Dividing in the usual way, we have $10.68 \div 13.02 = 0.8203$. In this case the two results agree to three places only, the last figure of 0.8206 being approximate, as frequently happens in using a four-place table.

Exercises. Multiplication and Division

Perform the following operations by logarithms, checking the results by multiplying or dividing in the usual way:

1. $12 \times 72.$
2. $2.4 \times 3.8.$
3. $3.76 \times 9.24.$
4. $76 \div 48.$
5. $252 \div 76.2.$
6. $0.45 \div 6.54.$
7. $52.6 \times 38.9.$
8. $0.624 \div 1.650.$
9. $24.25 \times 47.75.$

Using logarithms, perform the following operations:

10. $54.78 \times (-68.24).$
11. $(-6.105) \div (-12.36).$
12. $78.82 \times -92.88.$
13. $-21.98 \div 0.075.$
14. $(-0.645) \times (-28.73).$
15. $5.7 \times 6.3 \times 9.8.$
16. $12.56 \times 18.31 \times 41.65.$
17. $82.16 \times 42.75 \times 64.76.$
18. $3.142 \times 20.72 \times 16.34.$
19. $16.25 \times 18.56 \div 21.05.$

In Ex. 19 find the logarithm of the product of 16.25 and 18.56 first, and then subtract the logarithm of 21.05 from the result.

Using the Cologarithm. Since we may write $\frac{a}{b}$ as $a \cdot \frac{1}{b}$, we have

$$\log \frac{a}{b} = \log a + \log \frac{1}{b}.$$

We call $\log \frac{1}{b}$ the *cologarithm of* b, which is usually written "colog b." Since

$$\log \frac{1}{b} = \log 1 - \log b = 0 - \log b = -\log b,$$

we have $$\textbf{colog } b = -\log b.$$

For example, $\log \frac{1}{6} = \text{colog } 6 = -\log 6$. To avoid a negative mantissa we may write $-\log 6$ as $10.0000 - \log 6 - 10$. We then have

$$\begin{aligned} & 10.0000 - 10 \\ \log 6 = & \ 0.7782 \ \ - \\ \text{colog } 6 = & \ 9.2218 - 10 = \bar{1}.2218 \end{aligned}$$

When we begin to subtract from the first figure at the right of 10.0000, all the other figures become 9's. In practice, therefore, we find colog 6 by beginning at the left and mentally subtracting 0, 7, 7, 8 from 9, and 2 from 10, writing down only $9.2218 - 10$, or $\bar{1}.2218$.

The division formula on page 75 may now be written as

$$\log \frac{a}{b} = \log a + \text{colog } b\,;$$

that is, instead of subtracting the logarithm of the divisor, we may add its cologarithm, which, as shown above, can be easily found from the table.

Cologarithms are particularly convenient when the expression whose value we wish to find is at all complicated.

For example, in finding the value of $\frac{3.28 \times 16.54}{0.79 \times 21.28}$ we have

$$\begin{aligned} \log 3.28 &= 0.5159 \\ \log 16.54 &= 1.2186 \\ \text{colog } 0.79 &= 0.1024 \\ \text{colog } 21.28 &= \underline{8.6721 - 10} \\ & \quad 10.5090 - 10 = 0.5090 \end{aligned}$$

Since antilog $0.5090 = 3.228$, the value of the expression is thus found, by the simple process of addition, to be 3.228. It would obviously take considerable time to compute this result in the usual way.

Exercises. Multiplication and Division

Find the value of each of the following expressions:

1. $\dfrac{10.8 \times 12.5 \times 17.9}{35.7}$.

2. $\dfrac{325 \times 14 \times 1475}{360 \times 200}$.

3. $\dfrac{1146 \times 23 \times 92.75}{360 \times 300}$.

4. $\dfrac{18.96 \times 21.68 \times 35.08}{29.31 \times 17.37}$.

5. $\dfrac{24.78 \times 79.75 \times 109.60}{37.95 \times 45.91}$.

6. $\dfrac{78.87 \times 66.47 \times 71.59}{13.59 \times 98.35 \times 62.45}$.

7. Find by logarithms the area of a rectangular building lot that is 274.5 ft. long and 105.8 ft. wide; of a lot that is 327.8 ft. long and 238.2 ft. wide.

8. Find the area of a triangle which has a base of 35.9 cm. and a height of 17.8 cm.

Using the formula $A = \frac{1}{2}\,bh$, we have $A = \frac{1}{2} \times 35.9 \times 17.8$; that is, we must find the value either of $0.5 \times 35.9 \times 17.8$, or of 35.9×8.9.

9. Taking the value of π as 3.142, find the circumference of a circle with a diameter of 36.2 in.; of 52.67 ft.

Using the formula $C = \pi d$, we see that in the first case we have to find the value of 3.142×36.2 to four figures, which is as far as the result would be reliable from a four-place table.

10. Taking $7\frac{1}{2}$ gal. to the cubic foot, find the capacity in gallons of a rectangular tank 20.6 ft. long, 12.8 ft. wide, and 6.4 ft. deep, all measurements being inside dimensions.

11. Find the radius of a circle which has a circumference of 143 in.; of 25.69 ft.

From the formula $C = 2\,\pi r$, we have $r = \dfrac{C}{2\,\pi}$. That is, in the first case we have to find the value of $\dfrac{143}{2 \times 3.142}$, or of $\dfrac{143}{6.284}$.

12. Find the diameter of a circle which has a circumference of 19.5 in.; of 76.35 ft.

13. Find the radius of a cylinder with a volume of 265.2 cu. in. and a height of 15 in.

Finding a Power by Logarithms. The familiar algebraic law relating to the exponent of the power of a power is that

$$(a^m)^n = a^{mn}.$$

If, as on page 75, we take any positive number a and let l be its logarithm, we have

$$a = 10^l, \quad \text{or} \quad \log a = l.$$

If we now raise a to the nth power, we have

$$a^n = (10^l)^n = 10^{nl}.$$

Hence $$nl = \log a^n,$$

or $$\log a^n = n \log a.$$

Expressed in words, this formula shows that

The logarithm of a power of a number is the product of the logarithm and the exponent.

Finding a Root by Logarithms. In modern algebra the nth root of a is frequently expressed as $a^{\frac{1}{n}}$. The above law for the logarithm of a power is equally valid for fractional exponents; that is,

$$\log a^{\frac{1}{n}} = \frac{1}{n} \log a.$$

Expressed in words, this formula shows that

The logarithm of a root of a number is the quotient of the logarithm of the number divided by the index of the root.

That is, taking the number a with the logarithm l, as above, we have

$$a^{\frac{1}{n}} = (10^l)^{\frac{1}{n}} = 10^{\frac{l}{n}};$$

whence $$\log a^{\frac{1}{n}} = \frac{l}{n},$$

or $$\log a^{\frac{1}{n}} = \frac{\log a}{n}.$$

We thus see that by using logarithms it is about as easy to find a twelfth root, or any other root, as it is to find a square root. The results are, of course, only approximate, but so are the square roots given in the tables on pages 298–301.

Illustrative Examples. 1. Find the value of 1.05^6.

From the table, $\log 1.05 = 0.0212$,

and hence $\log 1.05^6 = 6 \times 0.0212 = 0.1272$.

Then $\text{antilog } 0.1272 = 1.340$.

That is, to four figures the value of 1.05^6 is 1.340.

2. Find the value of $\sqrt[5]{0.387}$, or of $0.387^{\frac{1}{5}}$.

We have $\log \sqrt[5]{0.387} = \dfrac{\log 0.387}{5} = \dfrac{\bar{1}.5877}{5} = \dfrac{4.5877 - 5}{5}$.

The last step in the above work was taken in order to have a negative part of the characteristic that will be exactly divisible by 5. Then dividing $4.5877 - 5$ by 5, we have

$$\log \sqrt[5]{0.387} = 0.9175 - 1.$$

Then $\text{antilog } \bar{1}.9175 = 0.8270$.

That is, to four decimal places the value of $\sqrt[5]{0.387}$ is 0.8270.

Exercises. Powers and Roots

Perform the following indicated operations by logarithms:

1. 62.4^2.
2. 37.84^3.
3. 25.32^4.
4. 6.075^{12}.
5. 0.00456^3.
6. $\sqrt{5}$.
7. $\sqrt[4]{7}$.
8. $\sqrt[6]{0.0003}$.
9. $45^{\frac{2}{3}}$.
10. $\sqrt[4]{38.72^3}$.
11. $\sqrt{2.86 \times 24.72}$.
12. $\sqrt[3]{0.62 \times 18.37}$.
13. $\sqrt{62.37} \div \sqrt[3]{3.142}$.
14. $0.3988^{\frac{1}{2}} \times 73.48^{\frac{2}{3}}$.
15. $0.000687^2 \div 0.4289^{\frac{2}{3}}$.

Find the value of each of the following expressions:

16. $\left(\dfrac{7.28 \times 8.27}{0.142 \times 18.56}\right)^2$.
17. $\sqrt{\dfrac{57.6 \times 82.73}{14.69 \times 28.75}}$.
18. $\sqrt[3]{\dfrac{82.1 \times 65.2 \times 98.4}{63.4 \times 32.7 \times 18.5}}$.
19. $\sqrt[4]{\dfrac{12.8 \times 34.6 \times 17.6}{16.9 \times 28.7 \times 32.8}}$.
20. $\left(\dfrac{18.56 \times 0.97}{1.65 \times 38.24}\right)^{\frac{1}{2}}$.
21. $\left(\dfrac{16.50 \times 1.02^2 \times 0.99}{2.75 \times 1.03^3 \times 0.70}\right)^{\frac{1}{2}}$.
22. $\left(\dfrac{28.76 \times 13.54 \times 34.67}{28.65 \times 83.07 \times 92.47}\right)^{\frac{2}{3}}$.
23. $\left(\dfrac{62.36 \times 43.27 \times 82.71}{18.37 \times 19.56 \times 56.54}\right)^{\frac{3}{4}}$.

Exercises. Using Logarithms

Perform the following indicated operations by logarithms:

1. 28×375.
2. 39×415.
3. -4.57×37.52.
4. -0.94×127.5.
5. 0.73×1.895.
6. $-114 \div 375$.
7. $98.75 \div 0.89$.
8. $172.8 \div 51.6$.
9. $53.75 \div 17.9$.
10. $0.97 \div 0.185$.
11. $-32.7 \div 12.94$.
12. $(-15)^4$.
13. 37.5^2.
14. -12.9^3.
15. 24.75^4.
16. $47.5^{\frac{1}{2}}$.
17. $83.74^{\frac{1}{3}}$.
18. $39.45^{\frac{3}{4}}$.
19. $\sqrt{49.49}$.
20. $\sqrt[3]{17.28}$.
21. $\sqrt[4]{593.6^3}$.
22. $\sqrt[5]{153.9^2}$.
23. $\sqrt{2.95 \times 13.75}$.
24. $\sqrt[3]{1.732 \times 1.414}$.
25. $\sqrt[4]{8 \times 753 \times 1.79}$.
26. $\sqrt{895.4} \times \sqrt[3]{64.2}$.
27. $0.693^{\frac{1}{2}} \times 956^{\frac{1}{3}}$.
28. $1.794^{\frac{2}{3}} \div 1.794^{\frac{1}{2}}$.
29. $3.298^{\frac{3}{4}} \times \sqrt{144.9}$.
30. $\frac{1}{2} \times \frac{28}{365} \times 755$.
31. $\frac{3}{4} \times 0.78^{\frac{1}{2}} \times 1.63^{\frac{1}{2}}$.
32. $29^{\frac{1}{3}} \times 89^{\frac{1}{2}} \times 92^{\frac{1}{6}}$.
33. $\sqrt{15} \times \sqrt[3]{17} \times \sqrt[4]{22}$.

34. From the formula $A = \pi r^2$, which gives the area of a circle, derive a formula for r. Find the radius of a circle having an area of 17.59 sq. in.; of 258.7 sq. ft.

When using logarithms, take the value of π as 3.142.

35. The formula for the area of a circle in terms of the diameter is $A = \frac{1}{4} \pi d^2$. Find the diameter of a circle which has an area of 28.15 sq. cm.; of 157.5 sq. m.

36. The formula $V = \frac{4}{3} \pi r^3$ gives the volume V of a sphere in terms of the radius r. Find the radius of a sphere which has a volume of 32.75 cu. in.; of 115.8 cu. ft.

37. The formula $V = \pi r^2 h$ gives the volume V of a circular cylinder with the radius r and the height h. What radius should be allowed in planning a cylindric gallon measure (231 cu. in.) which is to have a height of 14 in.?

38. The formula for the area of a regular octagon in terms of the side s is $A = 4.828\, s^2$. What is the area of a regular octagon that is 8.17 in. on a side?

39. The formula for the area of a regular hexagon in terms of the side s is $A = 2.598\, s^2$. What is the length of the side of a regular hexagon having an area of 32.25 sq. in.?

40. The formula $M = 0.00545\, l(d + d')(d - d')$ gives the number of cubic feet of metal in l feet of water pipe which has an outside diameter of d inches and an inside diameter of d' inches. Find the number of cubic feet of metal in a 20-foot length of water pipe having an outside diameter of 26 in. and an inside diameter of 20 in.

Find the values of $d + d'$ and $d - d'$ before applying logarithms.

41. If the pipe mentioned in Ex. 40 is made of cast iron weighing 450 lb. per cubic foot, what is the approximate weight in tons of 3 mi. of this pipe, allowing 8% extra of the total weight for the flanges which join the lengths?

42. The triangle formula $T = \sqrt{s(s-a)(s-b)(s-c)}$, in which $s = \frac{1}{2}(a + b + c)$, gives the area T of a triangle in terms of the sides a, b, and c. Find the area of a triangle in which the sides are 14.13 in., 27.17 in., and 26.39 in. long.

43. The distance d in feet passed over by a freely-falling body in t seconds is given by the formula $d = \frac{1}{2}\, gt^2$, where g is a constant with the value 32.16. Find the approximate time that it will take a body to fall 1150 ft.

44. From the horse-power formula $H = \dfrac{d^2 n}{2.534}$, which gives the horse power of an engine with n cylinders each d inches in diameter, find the horse power of an engine which has eight cylinders, each with a diameter of 3.075 in.

45. Using the formula $A = 0.7854\, d^2$, find the area of a circle which has a diameter of 12.75 in.

46. The formula for the area T of the total surface of a cube in terms of the volume V is $T = 6\, V^{\frac{2}{3}}$. Taking $7\frac{1}{2}$ gal. to the cubic foot, find the area of the total surface of a closed cubic container which has a capacity of 275 gal.

Exercises. Test on Chapter IV

Write the integers between which lie the logarithms of

1. 76.5. 2. 329.1. 3. 8375. 4. 8.1790. 5. 0.0275.

Show that the following statements are true:

6. $\log 1 + 3 \log 10 + \log 100 + \log 1000 + \log 0.001 = 5.$

7. $9 \log 1 + 8 \log 10{,}000 + \log 0.1 + \log 0.000001 = 25.$

Supply the missing parts of the following statements:

8. $\log ab =$ ____.

9. $\log$ ____ $= \log a - \log b.$

10. $\log$ ____ $= \log a + \operatorname{colog} b.$

11. $\log a^{\frac{m}{n}} =$ ____.

12. $\log a^n =$ ____.

13. $\log$ ____ $= n \log a.$

Using the table of logarithms on pages 314 and 315, complete each of the following statements:

14. $\log 28 =$ ____.

15. $\log 3.29 =$ ____.

16. $1.6485 = \log$ ____.

17. $\log 0.069 =$ ____.

18. $\log 11.56 =$ ____.

19. $\bar{2}.9485 = \log$ ____.

In each of the following cases find the value by logarithms:

20. $39 \times 57.$

21. $0.39 \times 5.7.$

22. $\dfrac{1.35 \times 17.53}{16.5}.$

23. $31.8 \times 45.7.$

24. $3.27 \times 21.57.$

25. $\dfrac{128.6 \times 218.4}{1728}.$

26. $\sqrt[3]{128.5}.$

27. $84.65^{\frac{2}{3}}.$

28. $\dfrac{3.142 \times 17.6^2}{144}.$

29. The velocity v in feet per second of a body that has fallen a distance of d feet is given by the formula $v = \sqrt{2\,gd}$, where g is a constant with the value 32.16. Find the velocity acquired by a body in falling 37.75 ft.

30. The time t in seconds for one oscillation of a pendulum l centimeters long is given by the formula $t = \pi \sqrt{\dfrac{l}{980}}$. Find the time of oscillation of a pendulum that is 21.8 cm. long; of one that is 43.6 cm. long.

Exercises. General Review

1. In a trade price list the cost C, in dollars per foot, of sewer pipe d inches in diameter is expressed by means of the formula $C = 0.004\ d^2 + 0.14$. Find the amount of an order for 3200 ft. of 12-inch sewer pipe.

Letting $x = 5$ and $y = -3$, find the value of

2. $(x + y)(x - y)$.

3. $x^2 + 2\ xy + y^2$.

4. $x^3 + 3\ x^2 + y$.

5. $x^3 + 3\ x^2y + 3\ xy^2 + y^3$.

Solve each of the following equations:

6. $x + 1.2 = 12.7$.

7. $y - 0.75 = 0.375$.

8. $2\ x - 5 = -3\ x^2$.

9. $\frac{1}{7}\ x - 2 = \frac{1}{5}\ x - 3.2$.

10. $\frac{2\ y}{3} = -\ y^2 + 40$.

11. $6\ x - 2.5 = 8.7$.

12. $\frac{1}{3}\ x + 4 = \frac{5}{3}\ x - 6$.

13. $\frac{3}{4}\ x - 2 = \frac{1}{2}\ x + 6$.

14. $2\ x + 7 = 11.5\ x + 3.2$.

15. $\frac{2\ x}{1.3} = 3.7$.

16. Write the meaning of the formula $C = s - (o + p)$, in which C represents the cost of an article, o the overhead charges, p the net profit, and s the selling price.

17. Solve the equation $2\ x^2 - 5\ x - 3 = 0$ graphically.

18. Solve by graphs the pair of equations $x^2 - 3 = 4\ y$ and $x + 2\ y = 6$, and check the roots by substituting them in both of the given equations.

19. Find the value of the expression 3.142×22.56^2.

Express the following products as powers:

20. $a^4 \cdot a^3$.

21. $x^5 \cdot x^7$.

22. $15^2 \times 15^3$.

23. $1.05^2 \times 1.05^5$.

24. $(1 + r)^5(1 + r)^4$.

25. $(x + y)^2(x + y)^6$.

Factor each of the following expressions:

26. $121\ x^4 - 9$.

27. $49\ a^2b^2 - 121$.

28. $p + prt$.

29. $4\ \pi R^2 - 4\ \pi r^2$.

30. $x^2 - 6\ x - 27$.

31. $x^2 - 13\ x + 22$.

CHAPTER V

ELEMENTARY SERIES

of little importance

Series. A sequence of terms in which each term after the first is formed from one or more preceding terms in accordance with some fixed relation is called a *series*.

For example, 1, 3, 5, 7 is a series in which each term after the first is formed by adding 2 to the preceding term. Similarly, 12, 8, 4, 0, $-4, \cdots$ and 3, 1, $\frac{1}{3}$, $\frac{1}{9}, \cdots$ are series.

Binomial Theorem. In the study of modern accounting practice, including such topics as depreciation, the amortization of loans, and the like, a very important series is the one formed by expanding the binomial $a + b$ to various powers. This series is also important in the study of genetics or of biology, in the study of the frequency of recurrence of phenomena (to be discussed in Chapter VI), and in other fields of science.

By actual multiplication we can easily show that

$$(a+b)^2 = a^2 + 2\,ab + b^2,$$
$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3,$$
$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4,$$
$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5,$$

and so on for any other integral power to which it is desired to raise $a + b$.

The law by means of which we can expand a binomial to any power, without the labor of actual multiplication, is called the *Binomial Theorem*.

The theorem, which is stated as a formula on page 86, is true, with certain restrictions, for negative and fractional exponents as well as for positive and integral exponents. In the applications that we shall make of it, however, we shall need only the latter.

The Binomial Formula. By studying the expansions given on page 85, and in particular such expansions as

$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5,$$

we can readily infer that a general formula for any power of $a+b$, which we may designate as the nth power, is

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n,$$

where the symbol 2!, which is called *factorial 2*, means $1 \cdot 2$, the symbol 3! means $1 \cdot 2 \cdot 3$, and so on.

Expressed in words, the above formula shows that

1. *The number of terms of the expansion is greater by 1 than the number of units in the exponent.*

That is, the number of terms is $n+1$. For example, in the expansion of $(a+b)^5$ the number of terms is $5+1$, or 6.

2. *The first and last terms of the expansion are respectively the first and last terms of the binomial raised to the power indicated by the exponent of the binomial.*

For example, in the expansion of $(a+b)^5$ the first and last terms are a^5 and b^5 respectively.

3. *The exponent of a in each succeeding term of the expansion decreases by 1, and the exponent of b, which may be regarded as 0 in the first term, increases by 1.*

For example, in the expansion of $(a+b)^5$ the successive terms contain a^5, a^4, a^3, $\cdots$, a^0 and b^0, b^1, b^2, $\cdots$, b^5 respectively.

4. *The coefficient of the first term is 1, the coefficient of the second term is the same as the exponent of the binomial, and the coefficient of any other term can be found by multiplying the coefficient of the preceding term by the exponent of a in that term and dividing the product by the exponent of b increased by 1.*

For example, the coefficient of the third term in $(a+b)^5$ is $4 \times 5 \div 2$, or 10, while that of the fourth term is $3 \times 10 \div 3$, or 10.

Using the Binomial Formula. The use of the Binomial Formula as a means of saving labor in writing expansions will be evident from the following examples:

1. Expand $(a+b)^7$.

Using the formula given on page 86, we have

$$(a+b)^7 = a^7 + 7\,a^6b + \frac{7\cdot 6}{1\cdot 2}\,a^5b^2 + \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}\,a^4b^3 + \frac{7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4}\,a^3b^4$$
$$+ \frac{7\cdot 6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5}\,a^2b^5 + \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\,ab^6 + b^7$$
$$= a^7 + 7\,a^6b + 21\,a^5b^2 + 35\,a^4b^3 + 35\,a^3b^4 + 21\,a^2b^5 + 7\,ab^6 + b^7.$$

The above work shows how the coefficients of the different terms of the expansion can be determined by substitution in the formula itself, but in practice it is frequently easier to use the fourth rule given on page 86. That is, from this rule we see at once in the above case that the coefficient of the third term is $6 \times 7 \div 2$, or 21; the coefficient of the fourth term is $5 \times 21 \div 3$, or 35; the coefficient of the fifth term is $35 \times 4 \div 4$, or 35; and similarly for the sixth and seventh terms.

2. Expand $(2\,x - 3\,y)^4$.

In this case we have $a = 2\,x$ and $b = -3\,y$. Substituting these values for a and b, and determining the coefficients as suggested in Ex. 1, we have

$$(2\,x - 3\,y)^4 = (2\,x)^4 + 4(2\,x)^3(-3\,y) + 6(2\,x)^2(-3\,y)^2$$
$$+ 4(2\,x)(-3\,y)^3 + (-3\,y)^4$$
$$= 16\,x^4 - 96\,x^3y + 216\,x^2y^2 - 216\,xy^3 + 81\,y^4.$$

As a simple check on the work, we may let $x = y = 1$. We then have

$$(2-3)^4 = (-1)^4 = 1,$$

and

$$16 - 96 + 216 - 216 + 81 = 1.$$

This does not, of course, check the exponents, but since they decrease or increase in a regular order, there is slight chance of failing to detect any errors made in writing them.

Exercises. The Binomial Theorem

Expand each of the following expressions:

1. $(x+y)^6$.	**4.** $(a+b)^8$.	**7.** $(a+3\,b)^4$.	**10.** $(3\,x-5\,y)^6$.
2. $(x-y)^6$.	**5.** $(a+1)^8$.	**8.** $(a-3\,b)^4$.	**11.** $(\frac{1}{2}\,x+3\,y)^5$.
3. $(x-y)^7$.	**6.** $(a-1)^8$.	**9.** $(2\,a-3)^4$.	**12.** $(1+0.04)^6$.

Formula for any Term. In the practical applications of the Binomial Theorem it is sometimes necessary to find some particular term. In order to avoid the necessity of writing the entire expansion as far as the particular term desired, let us see if we can discover any general relation between the nature of each term and its position in the expansion. For example, in the Binomial Formula, as given on page 86, we see that

$$\text{Term } 3 = \frac{n(n-1)}{2!}\, a^{n-2}b^2,$$

and

$$\text{Term } 4 = \frac{n(n-1)(n-2)}{3!}\, a^{n-3}b^3.$$

If we now represent any term, for example, the kth term, by the symbol T_k, it is easy to infer that a general formula for this term is

$$T_k = \frac{n(n-1)(n-2)\cdots \text{to } (k-1) \text{ factors}}{(k-1)!}\, a^{n-(k-1)}b^{(k-1)}.$$

Thus, as on page 86, the fourth term in the expansion of $(a+b)^5$ is

$$T_4 = \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}\, a^{5-3}b^{4-1} = 10\, a^2b^3.$$

Exercises. Finding Any Term

In each of the following cases find the specified terms:

1. $(x+y)^{10}$; 5th, 8th.
2. $(m-n)^9$; 4th, 7th.
3. $(2x+3y)^7$; 4th, 6th.
4. $(1+0.05)^{12}$; 7th, 9th.
5. $(2a-\frac{1}{3})^7$; 3d, 5th.
6. $(3x^2-\frac{3}{4})^9$; 2d, 7th.

7. In investment computations it is frequently necessary to find a value like that of 1.05^8 to a specified number of places. Find to the nearest 0.0001 the value of 1.05^6; that is, find the value of $(1+0.05)^6$, discarding any terms in the expansion which do not affect the fourth decimal place.

8. As in Ex. 7, find to five decimal places the value of 1.04^{12}; of 1.045^9; of 1.055^{15}.

Arithmetic Series. A series in which each term after the first is formed by the addition of a constant quantity to the preceding term is called an *arithmetic series.*

For example, 2, 4, 6, 8 is an arithmetic series of four terms, in which 2 is the constant added. Similarly, 6, 4, 2, 0, -2, -4, -6 is an arithmetic series of seven terms, in which -2 is added each time.

This type of series is also called an *arithmetic progression.*

The general form of an arithmetic series may be expressed as

$$a,\ a+d,\ a+2\,d,\ \cdots,\ l,$$

where d, the constant quantity added each time, is called the *common difference.* In work with series it is convenient to designate the first term by a, the last term by l, the number of terms by n, and the sum of all the terms by s. We often refer to a, d, l, n, and s as the *five elements* of a series.

In some books a capital S is used for the sum of all the terms.

When d is positive, the series $a,\ a+d,\ a+2\,d,\ \cdots,\ l$ is said to be an *increasing series.*

For example, the series 1, 2, 3, 4 is an increasing arithmetic series in which $a=1$, $d=1$, $l=4$, $n=4$, and $s=10$.

When d is negative, the series $a,\ a+d,\ a+2\,d,\ \cdots,\ l$ is said to be a *decreasing series.*

For example, the series 3, 1, -1, -3, -5 is a decreasing arithmetic series in which $a=3$, $d=-2$, $l=-5$, $n=5$, and $s=-5$.

The series $a,\ a+d,\ a+2\,d,\ \cdots,\ l$ may also be written in the reverse order as $l,\ l-d,\ l-2\,d,\ \cdots,\ a$.

For example, the increasing series 1, 3, 5, 7 may be reversed and written as the decreasing series 7, 5, 3, 1.

Arithmetic Means. The terms between any two specified terms of an arithmetic series are often called the *arithmetic means* between the two terms.

Thus, an example which asks for two arithmetic means between 1 and 7 requires us to find the second and third terms of an arithmetic series of four terms in which $a=1$ and $l=7$.

Last Term. It is not necessary to write out the whole of an arithmetic series when we wish to find any particular term of the series. For example, suppose that we wish to find the last term in the series of ten terms in which the first four terms are 1, 4, 7, and 10.

The second term is $1 + 3$, or $1 + 1 \cdot 3$;
the third term is $1 + 3 + 3$, or $1 + 2 \cdot 3$;
the fourth term is $1 + 3 \cdot 3$, and so on.

The tenth term of the series is therefore $1 + 9 \cdot 3$, or 28.

Since any term after the first term a in an arithmetic series is formed by adding the difference d to the preceding term, the total number of times that d is added to a in any term is 1 less than the number of that term in the series.

If we let the nth term be the last term l (that is, the last one in which we are at present interested), we therefore have

$$l = a + (n - 1)d.$$

For example, the fifth term of the series 1, 5, 9, $\cdots$ is $1 + (5 - 1) \cdot 4$, or 17, and the fiftieth term is $1 + (50 - 1) \cdot 4$, or 197.

Similarly, in the decreasing series 6, 4, 2, $\cdots$ the eighth term is easily seen to be $6 + (8 - 1)(-2)$, which is $6 + 7 \cdot (-2)$, or -8.

Exercises. Finding the Last Term

In each of the following series find the term specified:

1. $5, 11, 17, \cdots$; 11th.
2. $3, 8, 13, \cdots$; 14th.
3. $1, 1\frac{3}{4}, 2\frac{1}{2}, \cdots$; 12th.
4. $12, 4, -4, \cdots$; 14th.
5. $9, 6\frac{1}{2}, 4, \cdots$; 20th.
6. $-4, -9, -14, \cdots$; 12th.
7. $4x, 9x, 14x, \cdots$; 10th.
8. $3x, \frac{1}{2}x, -2x, \cdots$; 9th.
9. $n, 3n, 5n, \cdots$; kth.
10. $a^2, \frac{1}{4}a^2, -\frac{1}{2}a^2, \cdots$; nth.

11. If you save \$10 this week, \$15 next week, \$20 the week after, and so on, how much shall you save the sixth week?

12. What are the fourteenth and the fifteenth terms of the arithmetic series $x + y$, $2x + 3y$, $3x + 5y$, $\cdots$?

Sum of an Arithmetic Series. If we wish to find the sum s of an arithmetic series, such as the first six odd numbers, it is not necessary to add the terms in order to find their sum.

That is, by connecting the terms of the series in pairs, as shown below,

$$\overbrace{1 + \overbrace{3 + \overbrace{5 + 7}} + 9} + 11$$

we see that $1 + 11 = 3 + 9 = 5 + 7,$

and hence $s = 3 \cdot (1 + 11) = 36.$

Since 3 is half the number of terms, we see that the sum of this series may be represented by the expression $\frac{1}{2}\, n(a + l)$.

To find a general formula for the sum of an arithmetic series, we may express the sum s both as an increasing series and as a decreasing series, as follows:

$$s = a + (a + d) + \cdots + (l - d) + l,$$

and $$s = l + (l - d) + \cdots + (a + d) + a.$$

Adding these two expressions of n terms each, the result, which also contains n terms, is

$$2\,s = (a + l) + (a + l) + \cdots + (a + l) + (a + l),$$

or $$2\,s = n(a + l).$$

The general formula for the sum of all the terms of an arithmetic series is therefore

$$s = \tfrac{1}{2}\, n(a + l).$$

Exercises. Sum of an Arithmetic Series

In each of the following cases find the sum of the number of terms specified:

1. 3, 6, 9, $\cdots$; 15.
2. 4, 9, 14, $\cdots$; 15.
3. 1, 8, 15, $\cdots$; 8.
4. $\frac{1}{2}$, 1, $1\frac{1}{2}$, $\cdots$; 14.
5. 5, 7, 9, $\cdots$; 20.
6. 9, 8, 7, $\cdots$; 12.
7. 3, -3, -9, $\cdots$; 14.
8. -4, 3, 10, $\cdots$; 8.
9. $2\frac{1}{2}$, 5, $7\frac{1}{2}$, $\cdots$; 20.
10. $\frac{3}{4}$, $7\frac{3}{4}$, $14\frac{3}{4}$, $\cdots$; 6.
11. 8, 8, 8, $\cdots$; 8.
12. n, $3\,n$, $5\,n$, $\cdots$; k.

Other Formulas for Arithmetic Series. The two formulas which we have just developed, namely,

$$l = a + (n-1)d \tag{1}$$

and

$$s = \tfrac{1}{2}\, n(a+l), \tag{2}$$

are usually regarded as the basal formulas of arithmetic series. From these two formulas we can derive any others that are needed, as shown in the following examples:

1. Find a formula for s in terms of a, d, and n.

Substituting formula (1) in formula (2), we have

$$s = \tfrac{1}{2}\, n[a + a + (n-1)d]$$
$$= \tfrac{1}{2}\, n[2\,a + (n-1)d].$$

2. Find a formula for s in terms of d, n, and l.

From (1), $\qquad a = l - (n-1)d.$

Substituting this value of a in formula (2), we have

$$s = \tfrac{1}{2}\, n[l - (n-1)\,d + l]$$
$$= \tfrac{1}{2}\, n[2\,l - (n-1)\,d].$$

3. Find a formula for l in terms of a, n, and s.

From (2), $\qquad \dfrac{2\,s}{n} = a + l;$

whence $\qquad l = \dfrac{2\,s}{n} - a.$

4. Find a formula for d in terms of a, n, and s.

Using the formula found in Ex. 1, above, because it involves all four of the letters a, d, n, and s, we have

$$\frac{2\,s}{n} = 2\,a + (n-1)d;$$

whence $\qquad \dfrac{2\,s}{n} - 2\,a = (n-1)d,$

and $\qquad d = \dfrac{2(s - an)}{n(n-1)}.$

Formulas for Reference. In the list of formulas for arithmetic series given on the opposite page only the two basal formulas printed in heavy type need be memorized. The others may be referred to as necessary in solving the exercises.

I. Formulas for l	In Terms of
(1) $l = a + (n-1)d$	a, d, and n
(2) $l = \frac{1}{2}\left[-d \pm \sqrt{8\,ds + (2\,a-d)^2}\,\right]$	a, d, and s
(3) $l = \frac{2\,s}{n} - a$	a, n, and s
(4) $l = \frac{s}{n} + \frac{(n-1)d}{2}$	d, n, and s

II. Formulas for s	In Terms of
(1) $s = \frac{1}{2}\,n[2\,a + (n-1)d]$	a, d, and n
(2) $s = \frac{l+a}{2} + \frac{l^2 - a^2}{2\,d}$	a, d, and l
(3) $s = \frac{1}{2}\,n(a+l)$	a, n, and l
(4) $s = \frac{1}{2}\,n[2\,l - (n-1)d]$	d, n, and l

III. Formulas for a	In Terms of
(1) $a = l - (n-1)d$	d, n, and l
(2) $a = \frac{s}{n} - \frac{(n-1)d}{2}$	d, n, and s
(3) $a = \frac{1}{2}\left[d \pm \sqrt{(2\,l+d)^2 - 8\,ds}\right]$	d, l, and s
(4) $a = \frac{2\,s}{n} - l$	n, l, and s

IV. Formulas for d	In Terms of
(1) $d = \frac{l-a}{n-1}$	a, n, and l
(2) $d = \frac{2(s-an)}{n(n-1)}$	a, n, and s
(3) $d = \frac{l^2 - a^2}{2\,s - l - a}$	a, l, and s
(4) $d = \frac{2(nl-s)}{n(n-1)}$	n, l, and s

V. Formulas for n	In Terms of
(1) $n = \frac{l-a}{d} + 1$	a, d, and l
(2) $n = \frac{d - 2\,a \pm \sqrt{(2\,a-d)^2 + 8\,ds}}{2\,d}$	a, d, and s
(3) $n = \frac{2\,s}{a+l}$	a, l, and s
(4) $n = \frac{2\,l + d \pm \sqrt{(2\,l+d)^2 - 8\,ds}}{2\,d}$	d, l, and s

Exercises. Arithmetic Series

State which of the following series are arithmetic, and write the common difference in each of those cases:

1. 8, 10, 12, 14, $\cdots$.
2. $-12, -6, 0, 6, \cdots$.
3. $6, 2, \frac{2}{3}, \frac{2}{9}, \cdots$.
4. $2\frac{1}{2}, 0, -2\frac{1}{2}, -5, \cdots$.
5. $x, x^2, x^3, x^4, \cdots$.
6. $n, 4n, 7n, 10n, \cdots$.

Write the first seven terms of an arithmetic series in which

7. $a = 5$ and $d = 4$.
8. $a = 6$ and $d = -4$.
9. $a = -8$ and $d = 5$.
10. $a = 3x$ and $d = x - 3$.

11. Find a formula for a in terms of d, l, and s.

From the formula for l we have $n = \frac{l - a + d}{d}$, and from the formula for s we have $n = \frac{2s}{a + l}$. If we now equate these two formulas for n, we have the quadratic equation $a^2 - da = l^2 + ld - 2ds$. Solving this equation for a, we have formula III, (3) on page 93.

12. Find a formula for l in terms of a, d, and s.

Show how the following formulas on page 93 were derived:

13. I, (4). 14. II, (2). 15. III, (2). 16. IV, (4). 17. V, (4).

18. Insert five arithmetic means between 7 and 19.

Counting a and l, there will be seven terms. From the formula

$$l = a + (n - 1)d,$$

we then have $19 = 7 + (7 - 1)d$;

whence $d = 2$.

The series is therefore 7, 9, 11, 13, 15, 17, 19, and the required means are 9, 11, 13, 15, 17.

19. Insert seven arithmetic means between 12 and -12.

20. Insert one arithmetic mean between 3 and 41; between 7 and 11; between 9 and 18.

21. Show that a general formula for m, a single arithmetic mean between a and b, is $m = \frac{1}{2}(a + b)$.

Insert one arithmetic mean between

22. 6 and 18. **23.** a and $9\,b$. **24.** $5\,x$ and $-9\,x$.

Insert two arithmetic means between

25. 1 and 7. **26.** 25 and 67. **27.** a and $2\,a$.

Insert four arithmetic means between

28. -16 and 48. **30.** 0 and 18. **32.** 1.07 and 1.7.

29. -13 and 17. **31.** -26 and $1\frac{1}{9}$. **33.** $-3\sqrt{5}$ and $17\sqrt{5}$.

34. Which term of the series $28, 8, -12, \cdots$ is -72?

35. In the series $-30, -25, -20, \cdots$, which term is 80?

36. Which term of the series $\frac{2}{3}, 1, 1\frac{1}{3}, \cdots$ is $17\frac{1}{3}$?

In each of Exs. 37–41 find the sum of the series for each number of terms specified:

37. $3, 6, 9, \cdots$; 40 terms, 65 terms, 100 terms.

38. $\frac{1}{2}, 3\frac{3}{4}, 7, \cdots$; 30 terms, 50 terms, 80 terms.

39. $3+\sqrt{3}, \sqrt{3}, \sqrt{3}-3, \cdots$; 40 terms, 50 terms.

40. $x-y, x, x+y, \cdots$; 20 terms, 50 terms, 80 terms.

41. $\sqrt{a}-2, -2, -2-\sqrt{a}, \cdots$; 8 terms, k terms.

42. Find the sum of the first 100 positive integers.

43. What is the sum of the first 20 odd integers?

44. Find the sum of the first n odd integers, and show that when $n = 20$ the result agrees with the result of Ex. 43.

45. Beginning with 5, the numbers that are exactly divisible by 5 are $5, 10, 15, \cdots$. What is the sum of the first 20 numbers that are exactly divisible by 5?

46. Find the sum of the first 16 numbers, beginning with 9, that are exactly divisible by 9.

47. How many numbers, beginning with 1, must be taken in the series $1, 2, 3, \cdots$ in order to make the sum 1081?

Applications of Arithmetic Series. The following problems illustrate some of the simpler uses of arithmetic series:

1. If a secretary, starting at a salary of \$1000 a year, has her salary increased \$100 every year, what is the total amount that she will receive during 15 yr.?

The required answer is evidently the sum of an arithmetic series in which $d = 100$ and $n = 15$. Using formula II, (1), on page 93, since it involves the elements a, d, and n, we have

$$\begin{aligned} s &= \tfrac{1}{2}\, n[2\, a + (n-1)d] \\ &= \tfrac{1}{2} \cdot 15 \cdot (2000 + 14 \cdot 100) \\ &= 7\tfrac{1}{2} \cdot 3400 \\ &= 25{,}500. \end{aligned}$$

That is, the secretary will receive \$25,500 during the 15 yr.

2. If you save 50¢ the first week, 75¢ the second week, \$1 the third week, and so on, how much will you save in 1 yr.?

The required answer is evidently the sum of an arithmetic series in which $a = 0.50$, $d = 0.25$, and $n = 52$; that is, we customarily take 52 wk. to the year. Proceeding as in Ex. 1, we therefore have

$$\begin{aligned} s &= \tfrac{1}{2}\, n[2\, a + (n-1)d] \\ &= \tfrac{1}{2} \cdot 52 \cdot (1.00 + 51 \cdot 0.25) \\ &= 26 \cdot 13.75 \\ &= 357.50. \end{aligned}$$

That is, you will save \$357.50 during the year.

3. Suppose that at the beginning of each year a man invests \$100 in such a way that he receives simple interest at 6%. How much will he have at the end of 15 yr.?

The last investment bears interest for 1 yr. and amounts to \$106. The next to the last investment bears interest for 2 yr. and amounts to \$112, and so on to the first, which bears interest for 15 yr.; that is, we have an arithmetic series in which $a = 106$, $d = 6$, and $n = 15$. Hence

$$\begin{aligned} l &= a + (n-1)d \\ &= 106 + 14 \cdot 6 = 190, \end{aligned}$$

and

$$\begin{aligned} s &= \tfrac{1}{2}\, n(a + l) \\ &= \tfrac{1}{2} \cdot 15 \cdot (106 + 190) = 2220. \end{aligned}$$

The man will therefore have \$2220 at the end of the 15 yr.

Exercises. Arithmetic Series

1. It is proved in physics that a body falling freely travels approximately 16 ft. during the first second and approximately 32 ft. more in each succeeding second than it did in the one preceding. If a stone dropped from a stationary balloon reaches the ground in 10 sec., how far did it fall in the last second? How high is the balloon?

The distance traveled by a body during the first second is more nearly 16.08 ft., disregarding the resistance of the air, and the increase is 32.16 ft./sec., but the approximations given above are sufficiently close for ordinary purposes.

2. A baseball dropped from the top of the Washington Monument, a distance of 550 ft., was caught by an American League catcher. How fast was the ball falling when caught?

In Exs. 2 and 3 use the distances given in Ex. 1.

3. A body falling from a stationary balloon falls how far in the fifteenth second? in the nth second?

4. A dealer offers to sell a radio set on terms of \$5 down, \$5.50 at the end of 1 mo., \$6 at the end of 2 mo., and so on for twelve payments in addition to the initial one of \$5. How much in all will a purchaser pay for a set?

5. In a college savings club a student deposits \$1 the first week, \$1.25 the second week, \$1.50 the third week, and so on for the 36 wk. of the college year. What is the size of the deposit made in the last week, and what is the total amount deposited during the year?

6. A man buys a house for \$7000, paying \$2000 in cash and giving his note for \$5000 with interest at 6% a year. At the end of each year, in addition to the interest due, he pays \$500 on the principal. How many years will it take him to pay for the house?

7. How much will the man in Ex. 6 pay in all, including the cash payment, interest, and payments on the principal?

8. If a brake which reduces the speed by 320 revolutions per minute is applied to a wheel making 1600 revolutions per minute, how many revolutions will the wheel make before it stops?

9. A salesman who is receiving an annual salary of $2160, with no prospect of an increase, is offered a new position at an initial salary of $1920 and an annual increase of $120 for 12 yr. Find the difference in the amount of the annual salary for the two positions at the end of 5 yr.; of 12 yr.

10. In a factory in which the workmen receive 60¢ an hour during the first year, 65¢ an hour the second, and so on until they reach the maximum wage of 90¢ an hour, in which year will a workman reach the maximum wage?

11. A man bought a set of furniture for $800, paying $100 in cash and the rest in equal installments of $20 a month with interest at 6% on all unpaid balances. If the interest was all paid at once at the time of the last installment, what was the total amount paid, including the initial cash payment, the interest, and the monthly payments?

In actual practice the monthly payments are usually fixed at an amount that will take care of the interest for the entire transaction.

12. Some parts of the world use 24-hour time, counting midnight as 24 o'clock. If a clock strikes all the hours, how many strokes does it make each day?

13. In a potato race 20 potatoes are placed 3 ft. apart in a straight line leading to a basket that is 3 ft. from the first potato. Starting from the basket, the runner picks up one potato at a time and carries it back to the basket. Making no allowance for the distances that the runner can reach, how far does he run in completing the race?

14. Suppose that in Ex. 13 there are only half as many potatoes, but that the distances are twice as great. Will the total distance covered by a runner be the same as in Ex. 12? If not, what will it be?

Geometric Series. A series in which each succeeding term after the first is formed by multiplying the preceding term by a constant multiplier is called a *geometric series.*

For example, 3, 6, 12, 24 is an increasing geometric series of four terms in which the constant multiplier is 2. Similarly, 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$ is a decreasing geometric series of five terms in which the constant multiplier is $\frac{1}{3}$. The series 3, 3, 3, 3 is both a geometric series in which the constant multiplier is 1 and an arithmetic series in which the constant difference is 0.

If the constant multiplier is negative, we have a series that oscillates back and forth, sometimes increasing and sometimes decreasing, as in the case of $3, -6, 12, -24, 48, \cdots$, in which the constant multiplier is -2. A series of this kind is called an *oscillating series.*

Geometric series are also called *geometric progressions.*

The general form of a geometric series may be expressed as

$$a, \quad ar, \quad ar^2, \quad \cdots, \quad l,$$

in which a represents the first term, r the constant multiplier (called the *ratio*), and l the last term. The other elements of a geometric series are designated by the same letters which are used for arithmetic series; that is, the number of terms is designated by n, and the sum of all the terms by s.

As in arithmetic series, a capital S is sometimes used for the sum.

Geometric Means. The terms between any two specified terms of a geometric series are called the *geometric means* between the terms thus specified.

For example, in the series 3, 6, 12, 24 given above, 6 and 12 are geometric means between 3 and 24.

In the case of a single geometric mean we see that the ratio of the first term to the second is equal to the ratio of the second to the third; that is, we have

$$\frac{a}{ar} = \frac{ar}{ar^2}.$$

In other words, a single geometric mean is the mean proportional (see page 25) between the two specified terms. The mean proportional may, of course, be either positive or negative.

Exercises. Geometric Series

State which of the following are geometric series (giving the value of r in each case), which are arithmetic series (giving the value of d in each case), and which are neither:

1. $2, 4, 6, 8, \cdots$.
2. $1, 4, 16, 64, \cdots$.
3. $3, 15, 75, \cdots$.
4. $3, 10, 17, \cdots$.
5. $2, 10, 20, \cdots$.
6. $\frac{1}{a}, \frac{2}{a}, \frac{4}{a}, \cdots$.
7. $\frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots$.
8. $150, 75, 0, \cdots$.
9. $150, 75, 37\frac{1}{2}, \cdots$.
10. $16, -8, 4, -2, \cdots$.
11. $(x+y), (x+y)^2, (x+y)^3, \cdots$.
12. $\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{25}}, \sqrt{\frac{3}{125}}, \cdots$.
13. $\frac{1}{a^2}, \frac{2}{a^2}, \frac{3}{a^2}, \cdots$.
14. $\frac{10}{n}, \frac{8}{n}, \frac{6}{n}, \cdots$.

15. Under what conditions do a, b, and c form a geometric series? an arithmetic series?

16. Is $-1, -1, -1, -1$ an arithmetic series? a geometric series? an increasing series? a decreasing series?

Write the first four terms of the geometric series in which

17. $a = 3$ and $r = 3$.
18. $a = -2$ and $r = 3$.
19. $a = 2$ and $r = -2$.
20. $a = -2$ and $r = -2$.
21. $a = 48$ and $r = \frac{1}{4}$.
22. $a = \frac{1}{4}$ and $r = -\frac{1}{4}$.
23. $a = \frac{1}{3}$ and $r = 3$.
24. $a = 1 + x$ and $r = 1 - x$.

By continuing each of the following series as far as is necessary, find the term specified:

25. $1, 3, 9, \cdots$; 9th.
26. $64, 32, 16, \cdots$; 6th.
27. $2, -2, 2, \cdots$; 9th.
28. $\frac{1}{a}, -\frac{1}{2a}, \frac{1}{4a}, \cdots$; 6th.
29. $a, ar, ar^2, \cdots$; 8th.
30. $a, \frac{1}{3}a, \frac{1}{9}a, \cdots$; 10th.
31. $1, x^3, x^6, \cdots$; 6th.
32. $\frac{1}{n}, \frac{1}{n^2}, \frac{1}{n^3}, \cdots$; 8th.

Formula for the Last Term. Since any term after the first term a in a geometric series is formed by using r as a multiplier, the number of times that r is used as a multiplier of a in any term is 1 less than the number of that term in the series.

Thus, the first term a is multiplied by r once to obtain the second term ar, twice to obtain the third term ar^2, and $n-1$ times to obtain the nth term.

Designating the nth term as the last, a general formula for l, the last term, is therefore

$$l = ar^{n-1}.$$

For example, in the series 2, 6, 18, $\cdots$, the fourth term is $2 \cdot 3^3$, or 54.

Exercises. Finding the Last Term

In each of the following series find the specified term:

1. 1, 4, 16, $\cdots$; 8th.
2. 3, 12, 48, $\cdots$; 6th.
3. a, $3a$, $9a$, $\cdots$; 7th.
4. 2, -2, 2, $\cdots$; 20th.
5. $\frac{x}{y}, \frac{x^2}{y^3}, \frac{x^3}{y^5}, \cdots$; 10th.
6. 9, 3, 1, $\cdots$; 10th.
7. 16, -8, 4, $\cdots$; 10th.
8. 3, -6, 12, $\cdots$; 6th.
9. $x^2, x^3y, x^4y^2, \cdots$; 9th.
10. $\frac{1}{y}, \frac{2}{y^2}, \frac{4}{y^3}, \cdots$; 8th.

11. Find the last term in the series 1, $(x+1)$, $(x+1)^2$, $\cdots$ when $n = 10$; when $n = k$; when $n = 1$.

12. In the series $(1+m)$, $(1+m)^2$, $(1+m)^3$, $\cdots$, what is the eighth term?

13. By writing out the series 2, 10, 50, $\cdots$, find for what number of terms the last term will be 781,250.

14. By writing out the series 2, 5, 12.5, $\cdots$, find how many terms must be taken in order that the sum shall be 128.875.

15. Taking a and b as any two positive integers that are greater than 1, find how much greater (if any) is the nth term of the series a, ab, ab^2, $\cdots$ than the nth term of the arithmetic series a, $a+b$, $a+2b$, $\cdots$.

Finding Geometric Means. The method of finding geometric means is illustrated by the following examples:

1. Find three geometric means between 5 and 405.

Since there are to be three means, there must be five terms in all; that is, the problem reduces to finding the second, third, and fourth terms of a geometric series in which $a = 5$, $l = 405$, and $n = 5$. To do this, we must evidently find the value of r. From the formula

$$l = ar^{n-1},$$

we have $405 = 5r^4.$

Hence $r^4 = 81,$

and $r = \pm \sqrt[4]{81}$

$= \pm \sqrt{9}$

$= \pm 3.$

The series, therefore, either is 5, 15, 45, 135, 405, or it is 5, -15, 45, -135, 405, and hence the required means are either 15, 45, and 135, or -15, 45, and -135.

2. Find a single geometric mean between 7 and 252.

We might proceed as in Ex. 1, but since we know that a single geometric mean is the mean proportional between the two specified terms we may proceed as shown below.

Letting m represent the required single mean, we have

$$\frac{7}{m} = \frac{m}{252};$$

whence $m^2 = 7 \cdot 252,$

and $m = \pm \sqrt{1764}$

$= \pm 42.$

The required mean is therefore either 42 or -42; that is, the series may be either 7, 42, 252, or 7, -42, 252.

3. Find a general formula for inserting a single geometric mean between two specified terms of a geometric series.

Designating the first of the two terms as a and the second as l, and proceeding as in Ex. 2, we have

$$\frac{a}{m} = \frac{m}{l};$$

whence $m^2 = al,$

and $m = \pm \sqrt{al}.$

Exercises. Geometric Means

Insert one geometric mean between

1. 1 and x^8.
2. $\frac{1}{x}$ and $\frac{1}{x^8}$.
3. 1 and x^6.
4. $\frac{1}{2x}$ and $\frac{1}{32x^5}$.
5. 7 and 343.
6. $\frac{x^2}{y^2}$ and $\frac{x^6}{y^6}$.

Insert two geometric means between

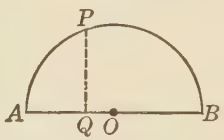
7. 2 and $\frac{27}{4}$.
8. $\frac{1}{3}$ and $\frac{1}{81}$.
9. $\frac{1}{16}$ and 4.
10. 5 and 135.
11. 1 and 64.
12. 1 and x^9.
13. 7 and 7.
14. x^8 and x^2.
15. n^2 and n^8.

Insert three geometric means between

16. 6 and 96.
17. 16 and 64.
18. 8 and 128.
19. 3 and 243.
20. a and b.
21. -9 and -144.

22. If the mean proportional (the geometric mean) between 3 and another number is 24, what is the other number?

23. It is proved in geometry that a perpendicular drawn from any point on a semicircle to the diameter is the mean proportional between the segments into which the diameter is divided; that is, in this figure, in which PQ is perpendicular to the diameter AB, PQ is the mean proportional between AQ and QB. If $AQ = 1$ in. and $QB = 2$ in., what is the length of PQ? What is the length of PQ when $AQ = 1$ in. and $QB = 3$ in.?



24. Draw a line that is $\sqrt{5}$ in. long.

We cannot, of course, express $\sqrt{5}$ precisely either by a common fraction or by a decimal. In a figure like the one in Ex. 23, above, we see, however, that $PQ = \sqrt{AQ \cdot QB}$, and hence if we construct the figure in such a way that $AQ = 1$ in. and $QB = 5$ in., PQ will be $\sqrt{5}$ in. long.

Draw lines of the following lengths:

25. $\sqrt{7}$ in. 26. $\sqrt{9}$ in. 27. $\sqrt{10}$ in. 28. $\sqrt{1\frac{1}{2}}$ in. 29. $\sqrt{1\frac{3}{4}}$ in.

Sum of a Geometric Series. The general formula for the sum of a geometric series may be derived by first stating the sum of the series in the usual form, as follows:

$$s = a + ar + ar^2 + \cdots + ar^{n-1}. \tag{1}$$

If we now multiply both sides of the equation by r, we have

$$rs = ar + ar^2 + \cdots + ar^{n-1} + ar^n. \tag{2}$$

Subtracting equation (1) from equation (2), we have

$$rs - s = ar^n - a;$$

whence $$(r-1)s = ar^n - a,$$

or $$s = \frac{ar^n - a}{r-1}.$$

Since $l = ar^{n-1}$, or $ar^n = lr$, we may substitute lr for ar^n in the above case, and thus have

$$s = \frac{lr - a}{r-1}.$$

We now have two convenient formulas for s, one in terms of a, n, and r, and the other in terms of a, l, and r.

For example, in a series in which $a = 3$, $n = 5$, and $r = 2$, we have

$$s = \frac{3 \cdot 2^5 - 3}{2-1} = 3 \cdot 32 - 3 = 93.$$

Similarly, if it is given that $a = 3$, $l = 48$, and $r = 2$, we have

$$s = \frac{48 \cdot 2 - 3}{2-1} = 96 - 3 = 93.$$

Exercises. Sum of a Geometric Series

1. Find the sum of eight terms of the series 2, 4, 8, $\cdots$.

2. In the series 3, 6, 12, $\cdots$ what is the sum of the first six terms?

3. In the series 64, 32, 16, $\cdots$ what is the sum of six terms? of ten terms? of n terms?

4. The sum of six terms of the series 3, 15, 75, $\cdots$ is either 11,618 or 11,718. Determine which it is.

In each of the following series find the sum of the number of terms specified:

5. $1, 4, 16, \cdots$; 9.
6. $1, \frac{1}{4}, \frac{1}{16}, \cdots$; 9.
7. $1, 5, 25, \cdots$; 8.
8. $1, \frac{1}{5}, \frac{1}{25}, \cdots$; 8.
9. $3, -6, 12, \cdots$; 10.
10. $3, -12, 48, \cdots$; 9.
11. $2, -6, 18, \cdots$; 8.
12. $2, 4, 8, \cdots$; 15.
13. $8, -8, 8, \cdots$; 16.
14. $\frac{1}{64}, \frac{1}{32}, \frac{1}{16}, \cdots$; 10.

15. If the first term of a geometric series is $\frac{2}{3}$, and the ratio is $1\frac{1}{2}$, what is the sum of seven terms?

16. Prove that $\log l = \log (a + rs - s) - \log r$.

17. In a geometric series in which $a = 6$, $r = \frac{1}{6}$, and $n = 6$, find the value of s. What is the last term?

18. Find the sum of k terms of the series $1, x^2, x^4, \cdots$.

19. Find the sum of six terms and also of n terms of the series $a - b, a^2 - b^2, a^3 - ab^2 + a^2b - b^3, \cdots$.

20. Find the sum of the first six terms of the geometric series in which the first two terms are a and ab. Check the result by letting $a = 2$ and $b = 3$.

21. If the first term of a geometric series is m and the third term is 1, what is the fourth term? What is the sum of the first four terms?

22. If the first three terms of a geometric series are 3, m, and $2m$, what is the value of m? What is the sum of the first five terms? What is the tenth term?

23. In a geometric series of six terms in which the last three terms are $6\frac{1}{2}$, $21\frac{1}{8}$, and $68\frac{21}{32}$ find the ratio, the first term, and the sum of the terms.

24. When the series $\frac{2}{5}, \frac{9}{10}, \cdots$ is an arithmetic series, what is the tenth term? What is the sum of ten terms? When the same series is a geometric series, what is the tenth term? What is the sum of ten terms?

Other Formulas for Geometric Series. We usually regard the three formulas

$$l = ar^{n-1}, \tag{1}$$

$$s = \frac{ar^n - a}{r - 1}, \tag{2}$$

and

$$s = \frac{lr - a}{r - 1} \tag{3}$$

as the basal formulas of geometric series. From them it is possible to derive others that are needed, as shown in the following typical examples:

1. Find a formula for l in terms of n, r, and s.

From (2), $s(r-1) = a(r^n - 1)$;

whence $a = \frac{s(r-1)}{r^n - 1}$.

Substituting this value of a in formula (1), we have

$$l = \frac{sr^{n-1}(r-1)}{r^n - 1}.$$

2. Find a formula for n in terms of a, l, and r.

Taking the logarithms of both sides of formula (1), which is

$$l = ar^{n-1},$$

we have $\log l = \log a + (n-1)\log r$.

Then $(n-1)\log r = \log l - \log a$,

and $n - 1 = \frac{\log l - \log a}{\log r}$,

or $n = \frac{\log l - \log a}{\log r} + 1$.

3. Find a formula for s in terms of r, l, and n.

From (1), $a = \frac{l}{r^{n-1}}$.

Substituting in (2),

$$s = \frac{\frac{lr^n}{r^{n-1}} - \frac{l}{r^{n-1}}}{r - 1}$$

$$= \frac{lr^n - l}{r^n - r^{n-1}}$$

$$= \frac{l(r^n - 1)}{r^n - r^{n-1}}.$$

Using the Formulas. The following examples illustrate the use of the basal formulas of geometric series and also of the formulas derived from them:

1. Find the last term of a geometric series in which $a = 7$, $n = 6$, and $r = 2$.

We have $$l = ar^{n-1};$$
whence $$l = 7 \cdot 2^5 = 7 \cdot 32 = 224.$$

2. Find the sum of the terms in Ex. 1.

We have $$s = \frac{ar^n - a}{r - 1};$$
whence $$s = \frac{7 \cdot 2^6 - 7}{2 - 1} = 7 \cdot 64 - 7 = 63 \cdot 7 = 441.$$

3. In a geometric series in which $a = 2$, $r = 3$, and $l = 4374$, what is the value of n?

From Ex. 2, page 106, $$n = \frac{\log l - \log a}{\log r} + 1.$$
By the table, $$\log 4374 = 3.6409$$
$$\log 2 = \underline{0.3010}$$
Subtracting, $$\log l - \log a = 3.3399$$
Then $$\log r = \log 3 = 0.4771.$$
Hence $$n = \frac{3.3399}{0.4771} + 1 = 7 + 1 = 8.$$
Check. $$l = ar^{n-1} = 2 \cdot 3^7 = 2 \cdot 2187 = 4374.$$

4. In Ex. 3 find the sum of the terms.

We have $$s = \frac{lr - a}{r - 1} = \frac{4374 \cdot 3 - 2}{3 - 1} = 6560.$$

5. Insert four geometric means between 2 and 6250.

Since $$l = ar^{n-1},$$
we have $$6250 = 2 \cdot r^5,$$
or $$r^5 = 3125.$$
Then $$5 \log r = \log 3125;$$
whence $$\log r = \frac{3.4949}{5} = 0.69898,$$
and $$\text{antilog } 0.6990 = 5.$$

Hence $r = 5$, and the required means are 10, 50, 250, and 1250.

Exercises. Geometric Series

1. Find the sum of ten terms of the series 4, 20, 100, · · ·.

2. Find the second term of the geometric series in which the first term is 7 and the third is 343.

3. Find the second and third terms of a geometric series in which the first term is 3 and the fourth is 81.

4. What is the sum of six terms of the geometric series $1.04, 1.04^2, 1.04^3, \cdots$.

5. Find a formula for r in terms of a, l, and s.

6. Find a formula for l in terms of a, r, and s.

7. Under certain conditions of temperature each of the bacteria in milk divides into two parts every 20 min. Assuming that at a given moment there are approximately 10,000 bacteria in 1 cu. cm. (cubic centimeter) of a sample of milk, and that the division begins 20 min. later, approximately how many bacteria will there be 5 hr. after the given moment?

8. In a country where the population is increasing in an approximate geometric series and is doubling itself every 50 yr., what is the approximate annual per cent of increase?

9. If 1, 2, and 5, respectively, are added to the three terms of an arithmetic series whose sum is 6, the three resulting numbers form a geometric series. What are the terms of the arithmetic series? of the geometric series?

10. Suppose that in a month containing 30 da. you start to save 2¢ the first day, 4¢ the next day, 6¢ the next day, and so on. How much would you be called upon to save on the last day of the month under this plan?

11. In Ex. 10, suppose that you wish to try the plan of saving 2¢ the first day, 4¢ the second day, 8¢ the third day, and so on. What is your estimate of the amount that you would be called upon to save on the last day of the month under this plan? Verify your estimate by using logarithms.

Infinitely Decreasing Geometric Series. In the geometric series 6, 3, $1\frac{1}{2}$, $\frac{3}{4}$, $\frac{3}{8}$, $\cdots$, since the ratio r is the positive proper fraction $\frac{1}{2}$, the terms are steadily decreasing. If we take n sufficiently large, the last term l can be made as small as we wish; that is, it will approach zero.

For example, the eighth term is $\frac{3}{64}$, the twelfth term is $\frac{3}{1024}$, and the hundredth term will obviously be very small indeed.

Since $ar^n < a$ and $r < 1$, we may avoid negative numbers by writing the formula for the sum of n terms as

$$s = \frac{a - ar^n}{1 - r}.$$

Dividing $a - ar^n$ by $1 - r$, we then have

$$s = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

If we now take n large enough, we make ar^n, and consequently the fraction $\frac{ar^n}{1 - r}$, approach zero. Hence, as n increases indefinitely, s approaches the fraction $\frac{a}{1 - r}$ as its limit. For brevity we usually write the formula for the sum of an infinite number of terms of a decreasing geometric series as

$$s = \frac{a}{1 - r}.$$

Repeating Decimals. A decimal like $0.5242424 \cdots$, in which a certain group of figures is continually repeated, is called a *repeating decimal.*

It is evident that the decimal $0.5242424 \cdots$ may be written as

$$0.5 + 0.024 + 0.00024 + \cdots,$$

where 0.024, 0.00024, $\cdots$ form an infinitely decreasing geometric series in which $a = 0.024$ and $r = 0.01$. We therefore have

$$s = \frac{a}{1 - r} = \frac{0.024}{1 - 0.01} = \frac{0.024}{0.99} = \frac{24}{990}.$$

Hence, to express the decimal as a common fraction, we have

$$0.5242424 \cdots = 0.5 + \tfrac{24}{990} = \tfrac{5}{10} + \tfrac{24}{990} = \tfrac{519}{990} = \tfrac{173}{330}.$$

Exercises. Geometric Series

Find the sum of each of the following infinitely decreasing geometric series:

1. $20, 5, 1\frac{1}{4}, \cdots$. **3.** $1, \frac{1}{2}, \frac{1}{4}, \cdots$. **5.** $4, -2, 1, -\frac{1}{2}, \cdots$.

2. $30, 3, 0.3, \cdots$. **4.** $1, \frac{1}{3}, \frac{1}{9}, \cdots$. **6.** $5, -1, \frac{1}{5}, -\frac{1}{25}, \cdots$.

In such cases it is understood that the word "sum" is used to mean the limit of the sum.

Express each of the following repeating decimals as a common fraction:

7. $0.333\cdots$. **9.** $0.232323\cdots$. **11.** $0.1666\cdots$.

8. $0.777\cdots$. **10.** $0.2325325\cdots$. **12.** $0.34125125\cdots$.

13. If you take half of a line 10 in. long, then half of what is left, and so on indefinitely, what is the sum of all the parts?

14. Suppose that a rubber ball is dropped from the top of a building 100 ft. high, and that on each successive rebound it reaches half its previous height. If the ball could continue to rebound indefinitely, how far would the ball travel?

15. After drawing a square $2k$ inches on a side, a student joins the midpoints of the sides, thus inscribing a second square within the first, and in the same way inscribes a third square within the second. If he could continue doing this indefinitely, what would be the total length of the lines drawn when $k = 8$?

16. If the third term of a geometric series is the square of the first term, what is the value of the ratio r? What is the sum of five terms?

17. If 3, m, and $3m + 12$ are consecutive terms of a geometric series, what is the value of m?

18. Find the common difference of the arithmetic series whose first term is 3 and whose second, fourth, and eighth terms form a geometric series.

Exercises. Test on Chapter V

Write the numbers 1–8 in a column, and after each number place the letter A if the corresponding series in the list below is arithmetic, G if it is geometric, and N if it is neither:

1. $5, 15, 45, \cdots$.
2. $x^2, 2xy, y^2, \cdots$.
3. $1, \frac{3}{4}, \frac{9}{16}, \cdots$.
4. $3x, 0, -3x, \cdots$.
5. $a^3, 3ab, 3ab^2, b^3, \cdots$.
6. $3a, a-5, a+5, \cdots$.
7. $1.04, 1.08, 1.12, \cdots$.
8. $3a, 3a+4, 3a+8, \cdots$.

9. Expand 1.05^8 to five terms; to six terms. If you wish to find the value to the nearest 0.00001, how many terms of this series do you need to take?

10. Show that a formula for the sum of n terms of an arithmetic series is $\log s = \log n + \log (a + l) + \operatorname{colog} 2$.

11. What total 5-year yield of corn will result from 10 original kernels, at 100 ears of 800 kernels each per bushel, if half of each planting matures, assuming $\frac{1}{20}$ of the preceding crop planted each year? (A live kernel yields 2 ears.)

12. If a flywheel $3\frac{1}{2}$ ft. in diameter is revolving at a speed of 60 revolutions per second just as the power is shut off, and if the speed then decreases $\frac{1}{2}$ revolution per second, how far will a point on the rim travel before the wheel stops?

13. Under favorable conditions the common house fly will mature and hatch her first litter 12 da. after being hatched, and will hatch a new litter each 12 da. thereafter. Assuming that there are 200 in each litter, equally divided as to sex, and assuming that no deaths occur, what will be the number of descendants of one female fly at the end of 36 da.? at the end of the 12 wk. of the summer season?

This assumption presupposes favorable breeding conditions throughout the season. The equal division with respect to sex is approximately correct.

The law of increase is expressed by $(a + b)^n$ in which $a = 1$ (the original female), $b = 100$ (the number of females in a litter), and n is the number of incubation periods.

Exercises. General Review

Perform the following indicated operations:

1. $9\,x - [4\,y - (5\,x + 3\,z)]$.
2. $6 - \frac{1}{5}(3 - 2\,a) - 4\,a$.
3. $7\,x - 5\,y + 5\,x - 3\,y$.
4. $\frac{1}{12}(7\,a + 3) - \frac{1}{4}(4\,a - 2)$.
5. $(m + n)^2$.
6. $c(x + 2\,y + 3)$.
7. $(3\,x + 2)(2\,x - 3)$.
8. $(a + b)(2 + x)$.

Solve each of the following equations:

9. $7\,x = 2.24$.
10. $\frac{1}{7}\,x = 2.24$.
11. $\frac{3}{5}\,x = \frac{1}{2}\,x + 2\frac{1}{2}$.
12. $x - 3(2 - 5\,x) = 32$.
13. $3(x + 15) + 5 = 2(2\,x - 3)$.
14. $2(x^2 - x) = 2\,x^2 - 16$.

By drawing their graphs, determine whether or not the equations in each of the following pairs are simultaneous:

15. $x^2 + y^2 = 13$, $x - y = -1$
16. $x^2 + y^2 = 13$, $xy = 6$
17. $x^2 - y^2 = 1$, $x - y = 0$

18. Arrange $4^{\frac{3}{2}}$, $34^{\frac{1}{2}}$, and 3^2 in order of magnitude, beginning with the smallest.

19. Find to two decimal places the value of $12^{\frac{3}{4}}$.

20. Find by logarithms the square root of 32.12; of 0.0744; of 842,700; of 29.62×38.75.

21. Show that the sum of the coefficients in the expansion of $(a + b)^n$ is 2^n.

If we let $a = b = 1$, the coefficients will not be changed, and the expansion will contain only the coefficients.

22. Many interesting stories have been written about geometric series; for example, there is the story about the invention of chess. The king to whom the game was shown asked the inventor to name his own reward, and the inventor suggested 1 grain of wheat for the first of the 64 squares, 2 grains for the second, 4 for the third, and so on. Find by means of logarithms the number of grains in the suggested reward.

CHAPTER VI

STATISTICS AND AVERAGES

Statistics. In Chapter III the student reviewed for simple cases the methods of constructing and interpreting graphs which represent statistics of various kinds. At that time no attempt was made to draw from the graphs any conclusions except those of a most general nature. The construction and interpretation of tables of statistics, and of the graphs based upon such tables, may almost be said to constitute a science in itself. In fact, we may very properly use the term "science of statistics," regarding it as including all questions relating not only to the statistics themselves but also to the graphic methods employed in representing them.

The chief purpose in the study of statistics is to enable the student to learn how the facts or tendencies established by masses of figures can be most clearly and truthfully set forth. As we have already seen in the simple cases previously studied, the use of graphs often makes it possible to show more clearly and more effectively the significance of a mass of statistics or to show how a single factor in a complicated situation may influence one or more others.

Statistics play an important part in the solution of our important social, governmental, and economic problems of today.

For example, a committee which is to study the relation between social conditions and crime would probably begin its work by the accumulation of a large amount of statistical material bearing upon the problem.

Sometimes, of course, the results of such statistical studies are negative, or they suggest the need of further study of the problem. Often, however, the results prove to be as dependable, for practical purposes, as a geometric formula.

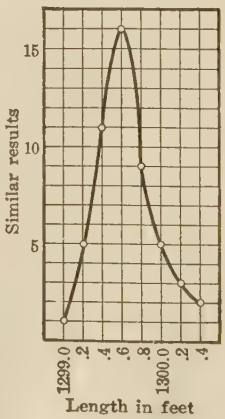
Observations and Measurement. In the use and interpretation of statistics relating to physical measurements of any sort a very important point is the accuracy of the measurements. Absolute precision in any kind of measurement is impossible, or as we often say, "*all measurements are only approximate.*"

Everyone is familiar with the wide variations in temperature observations during a very hot or a very cold spell; for example, two neighbors will often report results varying by 5°, probably due either to the inaccuracy of their thermometers or to carelessness in reading them.

In engineering, a frequent practice in securing results to a high degree of precision is to take all measurements several times and then to study the results statistically. For example, the following table shows the results of 52 measurements of the length of a proposed railroad tunnel, the number of times that the same results were obtained being recorded as follows:

Length in feet	1299.0	1299.2	1299.4	1299.6	1299.8	1300.0	1300.2	1300.4
Similar results	1	5	11	16	9	5	3	2

By plotting the points corresponding to these values we may draw the accompanying graph, which presents the results in a much more effective way than is possible with the above table.



Among the reasons for the variations in the measurements are failure to hold the tape always level, changes in the length of a steel tape due to changes in temperature, errors in reading or recording the measurements, and so on.

All that we can expect of any measurement is that it shall be accurate within certain *limits of error*.

For example, if we say that a measurement is correct to the nearest 0.1 ft. and record it as 10.3 ft., we mean that it lies between 10.25 ft. and 10.35 ft. Similarly, a length of 4.71 in., correct to the nearest 0.01 in., may be anywhere between 4.705 in. and 4.715 in., the limits of error being -0.005 in. and $+0.005$ in., and the total range of error being 0.01 in.

Statistical Classes. If in a series of statistical measurements involving a large number of items (sometimes hundreds or thousands) we consider the measures simply in the order in which they are determined, it is generally difficult to grasp the significance of the observations.

For example, it would be extremely difficult to form an opinion about the examination marks obtained by a class if we should consider them only in the order in which the marks are set down in the alphabetical list of students' names in the teacher's record book.

We therefore resort to classifying the measures in some definite manner so that the results may be more easily studied. The groups in which the measures are arranged are generally called *statistical classes*.

Scale and Range. We think of statistical classes as being distributed over a *scale*, which is usually represented, as we shall later see, by a graph or by a series of numbers.

The distance from the smallest measure to the largest one on a scale is called the *range*.

For example, in the scale 0 %–100 %, which is frequently used in marking examinations, the range is from 0 to 100. Even in marking examinations, however, the range need not be from 0 to 100; in fact, the choice of this range is sometimes unfortunate. For example, in the case of an algebra test containing 60 exercises, it is often more convenient to take the range as 60 and to say that a student who solves 55 exercises correctly has a score of 55. With scores computed to the nearest 1 %, three students who solved 54, 55, and 56 exercises correctly would be given scores of 90 %, 92 %, and 93 % respectively, thus giving to the second student a slight advantage in per cent which the other two do not have.

Methods of Grouping. The statistical treatment of the results of a series of observations depends upon how the classes are determined. For example, one method of grouping the students in a school would be to divide them according to their abilities into retarded, normal, and advanced classes. Another method would be to group the students in relation to the numerical marks which they receive during a certain period of time, — for example, the first half of the school year.

Exercises. Statistical Observations

In Exs. 1 and 2 state what seem to be the normal conclusions to be drawn from the data:

1. The following table, which covers the months of May and June for a period of 8 yr. in a large wheat-producing region, shows (1) the amount of rainfall, (2) the general range of the temperature, and (3) the character of the wheat crop:

Year	Rainfall	Temperature	Crop
1	5 in.	Above normal	Fair to poor
2	9 in.	Cool	Good
3	8 in.	Below normal	Good
4	4 in.	Normal	Poor
5	10 in.	Below normal	Very good
6	$4\frac{1}{2}$ in.	Cool	Poor
7	$8\frac{1}{2}$ in.	High	Fair to good
8	3 in.	Hot	Failure

2. The table given below shows the results of an examination of a number of varieties of potatoes for the following points of comparison: (1) color of skin, (2) character of eyes, (3) shape, (4) eating qualities, and (5) keeping qualities:

Varieties Examined	Skin	Eyes	Shape	Eating Qualities	Keeping Qualities
5	Pink	Shallow	Long round	Good	Poor
15	Pink	Deep	Short flat	Poor	Poor
20	Pink	Medium	Short round	Good	Good
20	White	Medium	Short flat	Poor	Good
15	White	Deep	Long flat	Poor	Poor
25	White	Shallow	Long round	Good	Good

State the limits of error and the range of error in each of the following measures:

3. 7.2 in. **6.** 8.2 mi. **9.** 2.4 sec. **12.** 4.2 cm.

4. 6.5 bu. **7.** 12.5%. **10.** 9.25 rd. **13.** 9.81 m.

5. 9.85 lb. **8.** 36.52 in. **11.** 5.481 mi. **14.** 10.43 g.

Class Interval. Before the results of a series of observations can be grouped in classes, as suggested on page 115, it is necessary to determine the interval on the scale that will best suit the given conditions. This interval is generally called the *class interval.*

For example, in classifying the results obtained on an examination we may compute the percentages to the nearest 1 % and give the mark A to all students whose standings are above 90 %, the mark B to all students whose standings are above 80 % but not above 90 %, and the mark C to all whose standings are above 70 % but not above 80 %. In this case the three class intervals are 91 %–100 %, 81 %–90 %, and 71 %–80 % respectively.

There are certain guiding principles or rules which are used in determining the number of class intervals. For example, it is usually desirable to have not more than 20 intervals, and to choose a size that will make the tabulation simple and rapid. If the number of units of the scale within the range is small, we often let each unit represent a class interval instead of trying to group the measures into smaller classes.

For example, in measuring the heights of boys of the same age it will be found that the range is comparatively small. In such a case we might therefore let 1 in. or 1 cm. be the class interval.

There is also the general rule that the class intervals should always be equal.

This rule, however, is often broken in the case of certain official publications in order to secure brevity.

Frequently the number of intervals can be determined only by experimenting with the data involved. In general, much of the skill in statistical work is acquired simply through practice in determining class intervals.

Class Frequency. The number of measures in each class is known as the *class frequency.*

For example, if in measuring the lengths of ears of corn we find that 20 ears are between 8 in. and 9 in. long, we say that the frequency for the interval 8 in.–9 in. is 20. Similarly, on page 114 the frequencies of the similar results are 1, 5, 11, 16, 9, 5, 3, and 2 respectively.

Class Limits. The classes into which the measures are grouped are separated by definite values of the variable. These values are known as the *class limits.*

As previously stated, the class intervals should, in general, be equal in order to make possible a better comparison of the measures in the different classes. For example, in a statistical study of the increases in the population of a city it would be inconvenient to have part of the data show annual increases and other parts the increases for 5-year periods or for periods such as 1 yr. 6 mo. or $2\frac{1}{2}$ yr.

There are various methods of defining the class limits, one of the most common of which is to state the limits themselves. For example, if we let the mark A represent an examination score above 90%, B a score above 80% but not above 90%, and C a score above 70% but not above 80%, the class limits are respectively 91, 100; 81, 90; and 71, 80.

Another method is to express the interval in terms of its midvalue; for example, in the case of the examination marks just mentioned the intervals might be designated as $75\frac{1}{2}$, $85\frac{1}{2}$, and $95\frac{1}{2}$, or as 75.5, 85.5, and 95.5.

The midvalues may be stated as integers and the class limits as decimals; for example, for the midvalues 30, 35, and 40 the class limits would be 27.5, 32.5; 32.5, 37.5; and 37.5, 42.5 respectively. It is evident that midvalues without fractions make calculation easier, but the fractional limits make the tabulation of the measures less convenient and less accurate.

A third method is to state the interval in words; for example, we might state that the interval is "from 71 to, but not including, 81" and so on.

The method of stating the class limits is not so important as the accuracy with which they are defined. Care should be taken that the classes are mutually exclusive and that no uncertainty can result. In accomplishing this it is often helpful to state the accuracy used in the measurements.

For example, if the heights of a group of men are measured to the nearest $\frac{1}{8}$ in., the class limits would be $65\frac{1}{16}$, $66\frac{1}{16}$; $66\frac{1}{16}$, $67\frac{1}{16}$; and so on. If the heights are measured only to the nearest $\frac{1}{4}$ in. in each case, the class limits would be $65\frac{7}{8}$, $66\frac{7}{8}$; $66\frac{7}{8}$, $67\frac{7}{8}$; and so on.

Summary. Before proceeding further in our study of statistics, it will be a good plan to summarize the important concepts that have been already discussed. This can best be done by considering a table of statistics like the one at the right, which was prepared in making a study of the abilities of a group of students in solving applied problems in a mathematics course. The abilities of the students were measured on a scale running from 20 to 89.99 inclusive, or as we usually say, from 20 to 90.

Scale	Class Interval	Class Frequency
90		
	80–89.99	6
80		
	70–79.99	21
70		
	60–69.99	40
60		
	50–59.99	66
50		
	40–49.99	42
40		
	30–39.99	20
30		
	20–29.99	8
20		

The results were grouped, in the seven class intervals shown in the table; that is, 80–89.99, 70–79.99, · · ·, 20–29.99. The class frequency for each interval was then recorded as shown.

The student will find that quantitative data tend to concentrate more or less at certain values of the variable. For example, in the 60–69.99 interval, which contains 40 students, it is possible that any number of students from 1 to 40 had a mark of 62. That is, each unit of the scale may be regarded as made up of 10 smaller units, and care should therefore be used in interpreting the grouping in such cases.

Frequency Table. Facts such as those given above are rarely observed or recorded in a systematic form at the beginning of the study of a problem. When the results are arranged in a tabular form like that shown above, we have what is known as a *frequency table*, or a *table of frequency distribution*.

That is, the table shows the frequency with which the measures are distributed over the several intervals. For example, the above table shows that out of 203 students only 8 were in the 20–29.99 interval, 20 were in the 30–39.99 interval, and so on.

The scale in the above table was inserted merely to aid the student in summarizing the new concepts. It is not an essential part of a frequency table, the essential parts being the column of class intervals and the column of class frequencies. Since frequency tables are coming into common use, it is desirable that students learn the general principles of arranging data in scientific form.

Constructing a Frequency Table. The following suggestions will be found helpful in constructing frequency tables:

1. *Study the data and find the range;*
2. *Decide upon the best number of class intervals;*
3. *Determine the most convenient class limits;*
4. *Record the results in two columns, as on page 119.*

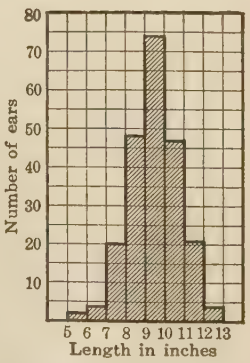
For example, suppose that in a group containing 220 ears of corn picked at random in a field the smallest ears are found to measure between 5 in. and 6 in. in length and the largest ears between 12 in. and 13 in.

Class Interval	Class Frequency
12–12.99	4
11–11.99	21
10–10.99	47
9– 9.99	74
8– 8.99	48
7– 7.99	20
6– 6.99	4
5– 5.99	2

Since the number of inches in this range is comparatively small, it will evidently be most convenient if we let the class interval be 1 in. This will give us eight classes for which the class limits are 5, 5.99; 6, 6.99; and so on.

We may then tabulate the frequencies with which the lengths fall within these limits, as shown in the second column.

Frequency Graph. Since a graph usually reveals the important aspects of a situation more readily than a table of statistics, let us now consider the accompanying graph, which was drawn from the data in the table above. Since the graph also shows the frequency with which the various lengths occur, it is called a *frequency graph.*



As we shall learn later, other types of graphs beside the bar graph may be used to represent a frequency distribution.

The graph tells us at a glance that there are many more ears between 9 in. and 10 in. in length than for any other size. In fact, there are more than in either the two preceding or the two succeeding classes together.

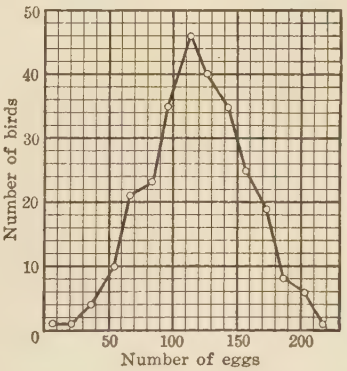
Frequency Polygon. In a study of the egg production in a flock of 275 barred Plymouth-Rock pullets, the birds which laid from 0 to 14 eggs were considered as a class laying an average of 7 eggs each; those which laid from 15 to 29 eggs were considered as a class laying an average of 22 eggs each; and so on.

That is, for convenience in making and recording the observations, the class interval was selected as 15.

The following table shows how the results were tabulated, the first line giving the classes (the average number of eggs laid), and the second the number of birds in each class:

Number of eggs	7	22	37	52	67	82	97	112	127	142	157	172	187	202	217
Number of birds	1	1	4	10	21	23	35	46	40	35	25	19	8	6	1

Instead of representing these statistics by a bar graph, such as was used on page 120, we may draw the accompanying broken-line graph by taking the number of eggs (the classes) on the x axis and the number of birds in each class on the y axis.



Since the graph is not a continuous curve, but jumps straight from one point to the next, it has the appearance of a multi-sided polygon. Such a graph is commonly called a *frequency polygon.*

Since, in the present discussion, we are more interested in the nature of the distribution than in the numerical sizes of the frequencies, it is evident that the broken-line graph is better suited to our needs than a bar graph.

From the graph the class of greatest frequency is easily seen to be the group of 46 birds which lay an average of 112 eggs apiece. This maximum class is also the central class on the scale; that is, there are seven classes of lesser frequency on each side of this maximum. On the left of the maximum, however, the frequencies decrease in a more irregular order than they do on the right.

Exercises. Frequency Graphs

1. Expand the binomial $(1 + 1)^8$, plot the numbers of the terms (1, 2, 3, · · ·) as abscissas and the values (1, 8, 28 · · ·) as ordinates, and draw a broken-line graph.

The result shows the position of the terms having the largest values. The principles of construction in this case are the same as those used in drawing a frequency polygon.

2. Toss four coins together sixteen times and record in a frequency table the number of heads and the number of tails. Plot a frequency polygon for the heads and one for the tails, using the same axes for both. Write a brief statement contrasting or comparing the polygons.

3. In a study of 300 ears of corn the lengths of the ears were measured to the nearest 0.5 in., and the frequencies with which the different lengths occurred were tabulated as follows:

Length	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5
Ears	1	1	2	2	4	9	9	11	20

Length	8.0	8.5	9.0	9.5	10.0	10.5	11.0	11.5	12.0
Ears	30	38	55	50	35	19	9	4	1

Plot a frequency polygon to represent these statistics.

4. In a group of 1334 draft mules withdrawn from the mining industry for service in the World War, the heights in inches were distributed as shown in the following frequency table:

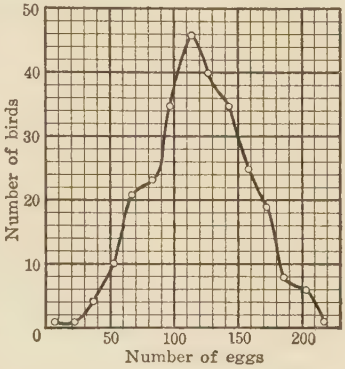
Height	59	60	61	62	63	64	65	66	67	68	69
Mules	4	3	4	2	8	19	38	80	137	202	224

Height	70	71	72	73	74	75	76	77	78	79	80
Mules	205	161	125	64	29	14	10	3	0	2	0

Plot a frequency polygon to represent these statistics.

Frequency Curve. If we should continually decrease the size of the class interval, and at the same time proportionally increase the number of measures, keeping the class frequencies finite, a frequency polygon would approach more and more closely a smooth curve. Such a curve is called a *frequency curve.*

That is, the polygon may be said to be smoothed into the curve.

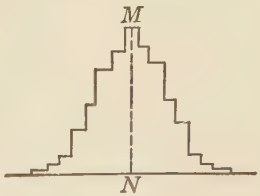


A frequency curve can be plotted by drawing a smooth curve through the vertices of a frequency polygon, as in the accompanying figure which was drawn from the frequency polygon shown on page 121. The curve may also be drawn through the midpoints of the tops of the bars in a bar graph.

Frequency Distribution. In statistical work, the manner in which the frequencies are distributed is often designated by means of the four following classifications:

1. Symmetric.
2. Moderately skewed.
3. J-shaped.
4. U-shaped.

Symmetric Distribution. In a symmetric distribution the class frequencies decrease uniformly from a central maximum point to zero on either side. In practical work, however, a perfectly symmetric distribution is rare.



The figure at the right is a frequency polygon showing the distribution of the heights, measured to the nearest 1 in., in a group of 25,878 adult American men. In this case the distribution approaches closely the symmetric, or ideal, type, and it would do so even more if $\frac{1}{4}$ in. had been taken for the class interval. An examination of the figure, however, will reveal that the distribution is not absolutely symmetric.

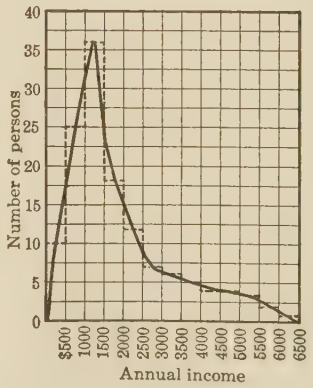
Skewed Distribution. In a moderately skewed distribution the curve tends away from the symmetric, or ideal, form; that is, the class frequencies decrease faster on one side of the maximum frequency than they do on the other. This type of distribution is the most common of all the smooth frequency curves.

Annual Income	Number of Persons
$0–$500	10
$500–$1000	25
$1000–$1500	36
$1500–$2000	18
$2000–$2500	12
$2500–$3000	7
$3000–$3500	6
$3500–$4000	5
$4000–$4500	4
$4500–$5000	4
$5000–$5500	3
$5500–$6000	2
$6000–$6500	1

As an example of this type, we may take the accompanying table, which shows the frequencies with which the several classes of incomes, $0–$500, $500–$1000, · · · , $6000–$6500, are distributed in a group of 133 persons.

From the table it is evident that the maximum frequency occurs in the $1000–$1500 interval. Instead of being symmetric, the curve which represents this distribution is skewed, as shown in the figure below, toward the side on which the larger incomes appear.

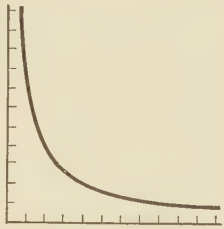
The meaning of the term "skewed" is clearly evident from the figure. As shown by the dotted lines, the curve is drawn by first locating the midpoints of the tops of the bars.



In actual practice it is frequently a matter of individual opinion in determining whether a curve that is nearly symmetric should be classified as a symmetric distribution or as a moderately skewed distribution.

Distribution is often affected by factors which upset the normal expectations; for example, in investigating the wages of carpenters we might naturally expect to find a few receiving very high wages and a few receiving very low wages. The agreements between employers and the unions, however, frequently result in the wages being the same for all.

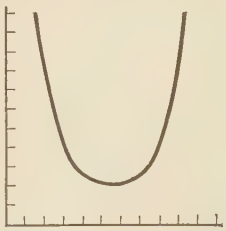
J-shaped Curve. The distribution curve here shown is a type not infrequently found when the conditions can be represented, either actually or approximately, by means of the equation $xy = k$. In statistical work, the curve which represents this type is known as the **J**-*shaped curve.*



This curve is found, for example, in plotting the frequency distribution of wealth in the United States, as represented by the annual income-tax returns. In this case the class frequency is a maximum at the lower end of the range; that is, a very large number of persons report the minimum taxable income. At the other end of the range the frequency is a minimum; that is, only a very few persons report a taxable income in the maximum class.

In such a case the distribution curve would seldom be as regular in shape as the one in the figure above, which shows the ideal curve represented by the equation $xy = k$. The tendency to decrease from a maximum at one end to a minumum at the other is, however, the same in both cases.

U-shaped Curve. The **U**-*shaped* type of distribution curve is here shown for an ideal case in which there is a maximum frequency at each end of the range, and the frequencies decrease uniformly to a minimum at the center.



An example of such a distribution would be found in a case where the results of an examination, marked on the scale A to E (A being the highest), show a maximum of A's and E's, a much smaller number of B's and D's, and only a few C's. The graph of this distribution would be a **U**-shaped curve like the one shown above.

Such a distribution of marks, it must be admitted, would be extremely unusual; in fact, this type is so rare that students will have little use for it. Most books on statistics include it, and it is therefore mentioned here for the sake of a complete discussion of the different types.

Exercises. Frequency Distributions

1. Draw a frequency curve showing the distribution of the lengths of the ears of corn in the case considered on page 120. Under what type of distribution should you classify this case?

2. After studying the frequency curve drawn in Ex. 1, what are your conclusions as to the relative number of long ears and short ears? Give reasons for your answer.

3. The following table shows the frequencies of the lengths in centimeters of 123 leaves picked at random from the same tree:

Length	Leaves	Length	Leaves
3–3.99	5	8–8.99	7
4–4.99	6	9–9.99	5
5–5.99	14	10–10.99	4
6–6.99	57	11–11.99	3
7–7.99	20	12–12.99	2

Draw a frequency bar graph, like the one on page 120, to represent these statistics.

4. The following numbers, which show the standings of the students in a class, are arranged in the order in which the students' names appear in the teacher's alphabetical list:

85	100	85	75	50	75	85	70	85	70	90
75	90	85	85	85	75	85	85	100	90	85
70	40	75	90	85	65	100	65	90	90	50

From the data construct a frequency table and from this table construct a frequency polygon.

5. The following table shows the frequency of height, in inches, of 430 school boys between 11 yr. and 12 yr. of age:

Height	60	59	58	57	56	55	54	53	52	51	50	49	48	47
Boys	3	4	12	22	36	54	79	78	59	35	29	12	4	3

Draw a frequency polygon to represent this distribution.

6. The table below shows that in a class of 115 students who took an algebra test containing 25 exercises, 1 student attempted 8 and only 8 exercises, 1 attempted 12 and only 12 exercises, and so on. It also shows that 1 student obtained only 1 correct solution, 1 obtained only 2 correct solutions, and so on.

Minimum number .	0	1	2	3	4	5	6	7	8	9	10	11	12
Attempted by . . .	0	0	0	0	0	0	0	0	1	0	0	0	1
Correctly solved by	0	1	1	0	0	0	3	4	5	9	14	13	7

Minimum number .	13	14	15	16	17	18	19	20	21	22	23	24	25
Attempted by . . .	1	4	4	10	13	13	8	22	13	18	6	1	0
Correctly solved by	11	15	8	10	3	7	1	2	0	0	1	0	0

Construct a curve showing the frequencies with which the different numbers of exercises were attempted and another showing the frequencies of the correct solutions.

7. The following table shows the results of counting the rays on each shell in a pail of scallop shells picked up at random:

Number of rays . .	15	16	17	18	19
Frequency	5	30	76	20	4

Draw a frequency curve to represent the above facts. State how you think the curve would look if several hundred shells were examined, and give the reason for your answer.

8. The following table shows the frequency of weight, measured in pounds, in a group of 1000 school boys 12 yr. of age:

Weight	49–56	56–63	63–70	70–77	77–84	84–91	91–98	98–105	105–112
Boys	4	24	118	233	273	221	79	36	12

Construct a frequency polygon showing this distribution and write a brief statement of what the graph reveals.

Theoretical Frequency Curve. As stated on page 123, an actual case of a frequency curve that is perfectly symmetric with respect to a vertical axis is rare. Such a curve is therefore known as a *theoretical frequency curve.*

The curve is also called an *error curve,* a *probability curve,* a *normal distribution curve,* and a *bell-shaped frequency curve.*

As an illustration of this type of distribution, we may take the accompanying table, which shows the theoretical frequencies of the hits in firing series of 5 shots each at a bull's eye.

Hits	Frequency
5	1
4	5
3	10
2	10
1	5
0	1

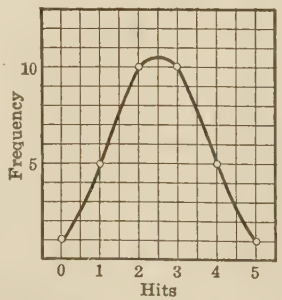
These frequencies are determined by examining the possible ways in which the different numbers of hits from 5 to 0 can be made. There is obviously only one way of making 5 hits; that is, every shot must be a hit. There is also only one way of making 0 hits (5 misses).

In the case of 4 hits, any one of the 5 shots may be a miss, and hence there are five possible ways of making this score. The same frequency applies to the case of 1 hit (4 misses).

In the case of 3 hits (2 misses), we may let *h* represent a hit and 0 a miss. If in the first series of 5 shots the first two are misses and the next three are hits, the score would be represented by the first column shown below. The remaining possibilities are shown in the other columns.

0	0	0	0	*h*	*h*	*h*	*h*	*h*	*h*
0	*h*	*h*	*h*	0	0	0	*h*	*h*	*h*
h	0	*h*	*h*	0	*h*	*h*	0	0	*h*
h	*h*	0	*h*	*h*	0	*h*	0	*h*	0
h	*h*	*h*	0	*h*	*h*	0	*h*	0	0

Since there are ten columns, the frequency for either 3 hits and 2 misses or 2 hits and 3 misses is evidently 10.



If we plot the curve representing this distribution, we shall find, as shown in the figure above, that it is symmetric.

It does not follow, of course, that in any series of 5 shots, or even in several series, the hits will occur with the frequencies stated. If a sufficient number of cases are observed, however, the distribution curve will be found to approximate very closely the curve shown above.

Theoretical Distribution. In the case discussed on page 128 it should be observed that the frequencies 1, 5, 10, 10, 5, 1 are the coefficients of the expansion of $(a + b)^5$, and that the total number of cases is 32, which is the sum of the coefficients. That is, the distribution tends to follow the law expressed by the expansion of $(1 + 1)^n$ when $n = 5$.

Thus, if a series of 20 shots is repeated a large number of times, the distribution curve of the hits will tend to approximate the curve whose ordinates are the successive terms that can be found by expanding the binominal $(1 + 1)^{20}$ to $1 + 20 + 190 + \cdots + 190 + 20 + 1$.

If there are two conditions affecting the recurrence of the phenomena, as in determining the frequency of the ace-deuce combination in throwing dice, the distribution will tend to follow the expansion of $(1 + 2)^n$; if there are three conditions, the distribution will tend to follow the expansion of $(1 + 3)^n$, and so on. It is apparent that the curves representing these distributions are skewed, and that the skewness increases as the second term of the binomial increases.

Exercises. Recurring Phenomena

1. Test the above discussion with respect to the expansion of $(1 + 1)^n$ by tossing five coins 32 times and arranging the results in classes according to the number of heads that turn up each time. Plot and compare the frequency curve with the curve representing the terms of $(1 + 1)^5$.

2. Throw four dice 16 times, recording the number of aces and deuces that turn up at each throw. Arrange the results in a frequency table and compare them with the terms of $(1 + 2)^4$. Plot both the statistical curve and the theoretical curve on the same axes, and see if there is any tendency to agree.

3. Throw four dice 256 times, recording the number of aces, deuces, and threes that turn up. Arrange the results in a frequency table and compare them with the terms of $(1 + 3)^4$. Plot both curves, as in Ex. 2.

Limitation of Statistics. There is a common saying among non-scientific people that anything can be proved by statistics. This saying is due to the fact that the public has often been deceived by figures which have been dishonestly manipulated or which have not been presented scientifically. Not all such cases, however, represent deliberate intent to deceive, since a person untrained in statistical methods is almost certain to make mistakes at some point in the work. Neither are all the mistakes made in statistical studies due to these two causes; even the careful student can lapse into unscientific methods of handling data.

In examining statistics, the student should always find out, if possible, (1) the reliability and training of those who gathered the facts, (2) how and when the facts were gathered, (3) to what extent the results have been submitted to the critical judgment of experts, and (4) to what extent similar studies show substantiating results.

Law of Statistical Regularity. If we wish to find the average height of a 10-year old boy in New York City, it is not necessary to measure and record the height of every boy of this age in the city. In fact, if we should measure a few boys chosen at random in several places throughout the city, we should probably obtain a result that would be very close to the exact average for the entire group.

In statistical work the following principle, which is known as the *Law of Statistical Regularity,* is widely used:

If a reasonable number of individual cases are chosen at random from among a very large group, they are almost sure, on the average, to possess the same characteristics as the larger groups.

In other words, the larger the group that is studied, the greater are the chances of obtaining a fair sample of the entire group by a random selection. Much of the work of the Census Bureau and of the statistical departments of the large insurance companies is based upon this principle.

Continuous Series. As we have already seen, in drawing graphs to represent changes in temperature, in population, in the heights or weights of children, and the like, we very frequently use a smooth curve. Such a curve would be the natural one to use in representing the facts which are given by the following record of a patient's temperature, as kept by the nurse:

Date	8 A.M.	6 P.M.	Date	8 A.M.	6 P.M.
May 1	102.4°	103.2°	May 4	100.6°	101.2°
May 2	102.8°	103.6°	May 5	99.4°	99.8°
May 3	102.6°	103.0°	May 6	98.6°	98.6°

In such a case, extreme variations between any two successive observations would not be possible, and it is therefore reasonable to assume that the patient's temperature changed gradually from one observation to the next. Such a collection of items is known as a *continuous series.*

For example, from the above table the physician might reasonably assume that the patient's temperature was approximately 103.2° at 1 P.M. on May 2.

Discrete Series. Suppose, however, that we wish to make a study of accidents from the accompanying table, which shows the number of killed and injured in Fourth of July accidents during a recent period of six years.

Year	Killed	Injured	Total
1	215	5092	5307
2	131	2792	2923
3	57	1546	1603
4	41	945	986
5	32	1163	1195
6	40	1506	1546

Although a few of the accidents due to the Fourth of July might actually occur on July 3 or on July 5, it is evident that a continuous curve drawn from the data in the table would show nothing with respect to the accident rate for the rest of the year. A collection of items like this is called a *discrete series,* or a *broken series.*

In representing the facts in this case, the graph, which preferably should be of the bar type, should be clearly labeled in order to show that the statistics relate only to July 4 of each year.

Exercises. Review

1. What is meant by saying that the range of error in measuring an object to the nearest 0.01 in. is 0.01 in.?

2. If the length, width, and depth of a rectangular tank are each measured correct to the nearest 0.1 ft., is it correct to give the volume of the tank to the nearest 0.001 cu. ft.? Give the reasons for your answer.

3. Name three tests that you would try to apply in examining the results of a statistical investigation.

4. Explain why in studying a large group with reference to some common characteristic it is not necessary to take separate measures of each item in the group.

5. What is meant by the statement that human characteristics tend to distribute themselves in accordance with a theoretical frequency curve?

6. Explain why a record of wages paid in a factory is likely to be a discrete series.

7. Do the results in throwing dice illustrate a continuous or a discrete series?

8. What type of series is represented by a day's record of the pressures shown on the gauge in a factory boiler room?

9. Discuss the advantages of a frequency polygon (or curve) over a frequency table.

10. What is the real purpose of organizing a table of facts into a frequency distribution?

11. If possible, find one or more illustrations of inaccurate or misleading graphical representations of statistics, and write a brief statement of the reasons why the type of representation used is not ideal.

Such examples may occur in current newspapers or magazines, particularly in cases where attempts are made to compare quantities by means of graphs in which it is necessary to compare the areas or the volumes of two or more geometric figures drawn to scale.

Averages. If a lecturer wished to give a group of farmers some idea of the relative lengths of the ears of corn grown in a crop experiment, he would scarcely attempt to read a list of the lengths of all the ears harvested from even a comparatively small field. Instead, he would probably tell them that the "average length" of ear was so many inches, for example, $9\frac{1}{2}$ in. His listeners would regard this as a fair sample of the yield which they could use in making comparisons.

This concept of "average" as indicating a fair sample or measure of some characteristic common to a large group is widely used in ordinary conversation, and in our work up to this point we have frequently used the word in this sense. In statistical work, however, it is necessary to be more precise and to know in what manner the average of a group is computed if we are not to be misled in drawing conclusions. We shall therefore consider first the nature and methods of computing each of the following kinds of averages:

1. The arithmetic mean, or the arithmetic average.
2. The weighted arithmetic mean.
3. The mode.
4. The median.
5. The geometric mean.

Since each of the five terms listed above is a special kind of average, it is preferable to use the expression "arithmetic mean" instead of "arithmetic average" for the first type.

Arithmetic Mean. As is well known from elementary arithmetic, the *arithmetic mean* (average) is simply the sum of all the items divided by the number of items.

For example, if 8 students obtain marks of 85 %, 76 %, 91 %, 83 %, 85 %, 81 %, 79 %, and 76 % respectively, on an examination, the arithmetic mean is found by dividing the sum of these per cents by 8; that is, we have

$$\tfrac{1}{8} \times (85 + 76 + 91 + 83 + 85 + 81 + 79 + 76) = \tfrac{1}{8} \times 656 = 82.$$

The arithmetic mean in this case is therefore 82 %, which shows that if we consider the ability of the class as a whole and divide it into 8 equal parts, the result is measured by a mark of 82 %.

Weighted Arithmetic Mean. In the discussion of frequency tables on page 120 the accompanying table was used as an illustration. In this case we have 4 ears whose length is between 12 in. and 12.99 in., 21 ears whose length is between 11 in. and 11.99 in., and so on, the class frequencies showing the number of items in each class. For convenience, let us now take the midpoint of each class interval to represent the size of the class; that is, we assume that in this case we have 4 ears 12.5 in. long, 21 ears 11.5 in. long, and so on.

Class Interval	Class Frequency
12–12.99	4
11–11.99	21
10–10.99	47
9– 9.99	74
8– 8.99	48
7– 7.99	20
6– 6.99	4
5– 5.99	2

Actually, of course, 18 ears in the second group might have been 11.8 in. long and 3 ears might have been 11.2 in. long, but in dealing with large groups it is obvious that errors of this sort tend to balance each other.

In computing the arithmetic mean length of ear it is evident that we must first find the total of the measures in each class. In finding this total, as shown by the work at the right, we simply

Multiply each measure by its frequency before adding.

Adding these results and dividing by 220, the number of items, we find that the arithmetic mean length of ear is approximately 9.5 in.

$$\begin{array}{r} 4 \times 12.5 = 50.0 \\ 21 \times 11.5 = 241.5 \\ 47 \times 10.5 = 493.5 \\ 74 \times 9.5 = 703.0 \\ 48 \times 8.5 = 408.0 \\ 20 \times 7.5 = 150.0 \\ 4 \times 6.5 = 26.0 \\ 2 \times 5.5 = 11.0 \\ \hline 220\overline{)2083.0} \\ 9.46\ldots \end{array}$$

The result in this case is, for all practical purposes, the same as if we had measured each of the 220 ears to the nearest $\frac{1}{2}$ in., added the 220 items thus recorded, and divided by 220. The amount of computation involved, however, is obviously much less.

An arithmetic mean computed in this way is called the *weighted arithmetic mean.* Except for the assumption of the midpoint as the measure of each class, however, there is no fundamental difference between it and the arithmetic mean.

Exercises. Arithmetic Mean

1. Find the arithmetic mean daily production per cow in a herd of 20 cows where the daily milk production is as follows:

Cow . . .	1	2	3	4	5	6	7	8	9	10
Pounds . .	30	22	20	27	25	30	30	27	25	30

Cow . . .	11	12	13	14	15	16	17	18	19	20
Pounds . .	22	22	22	25	20	27	27	30	25	22

2. In Ex. 1 make a frequency table showing the number of cows producing each of the specified weights, and compute the weighted arithmetic mean daily production per cow.

3. In Ex. 2, what is the advantage of using the weighted arithmetic mean? Why should each production be multiplied by the number of cows producing that amount of milk?

4. The following table gives statistics relating to the weights per bushel of the potatoes which were grown on an acre of land:

Pounds per bushel . . .	56	57	58	60	62	63
Number of bushels . . .	25	10	30	20	40	25

Find the weighted arithmetic mean weight per bushel.

5. The following table shows the frequency of weight, measured in pounds, in a group of 1000 school boys 13 yr. of age:

Weight	63–70	70–77	77–84	84–91	91–98	98–105	105–112	112–119	119–126
Frequency	44	100	225	256	187	112	43	17	16

Find the weighted arithmetic mean weight for this group.

6. Plot the frequency curve from the data in Ex. 5 and state whether or not there is a tendency to symmetry about the mean as an axis. Explain your answer.

Disadvantages of the Mean. Although the arithmetic mean is easily calculated, there are several objections to its use in certain cases. Many people think that the arithmetic mean, or the "average" as it is usually called in ordinary conversation, indicates the measure or factor of the most common occurrence in a given situation. For example, we often hear of the "average student," which supposedly means the student who spends an average amount of time on his studies, receives an average mark in each of his courses, is of average age and of average height, and so on. Of course, no one person meeting all these conditions exists, so that the ordinary use of the word "average" is frequently misleading.

As another example, let us take the accompanying frequency table, which represents a hypothetical case of a collection taken up in a group of 90 persons.

Amount Contributed	Number of Contributions
$0.00	2
0.01	23
0.05	42
0.10	13
0.25	9
50.00	1

In this case the mean contribution is approximately 62¢, but it is evident from the table that only one person gave as much as this, and that no one gave exactly this amount. The objection to the use of the arithmetic mean in this case is that it gives too much emphasis to the single large contribution of $50.

Similarly, to say that the mean weekly earnings of 10 men in an office are $40 will be misleading if two of them are receiving $100 per week, since the mean for the others will then be $25 per week.

Furthermore, in a frequency table or graph the arithmetic mean often occurs at a point where no item exists, and hence it cannot be easily located.

For example, in the above table the mean contribution is 62¢, but no one gave precisely this amount. Similarly, an investigation might disclose that in a certain community the average family has 4.89 members, although a family of precisely this size is, of course, impossible.

For these reasons it is often desirable to use other measures to show the central tendency of a distribution.

Mode. In the graph on page 120 the longest bar represents the class interval 9–9.99; that is, ears of corn varying in length from 9 in. to 9.99 in. are the ones that occur most frequently. The same fact is, of course, also shown by the frequency table at the top of the same page. In a distribution, the value which occurs most frequently is called the *mode*. In other words, the mode is represented by the class corresponding to the largest number in a frequency table, to the longest bar in a bar graph, or to the maximum point of a frequency curve.

For example, in the table on page 136 the mode is 5¢; in the bar graph on page 120 the mode may be taken as 9.5 in., the midpoint of the class interval 9–9.99; and in the frequency curve on page 124 the mode is $1250, the midpoint of the $1000–$1500 class.

We thus see that the mode rather than the mean is the measure which describes the most usual occurrence, — the "common thing." That is, instead of speaking of the "average student," as on page 136, it is more precise to speak of the "modal student" as the one who spends the most common amount of time on his studies, receives the most common marks, and so on.

Of course, as stated on page 136, no such student actually exists, because if he possessed one or more of the common characteristics, he would be almost certain to differ from the common type in others.

Another advantage of the mode over the mean is that the former cannot occur at some approximate point within a class. Also, it can generally be determined at sight from a frequency table or a frequency graph, so that in this sense it may be regarded as an *inspectional average*.

For example, in the case considered on page 136 it is evident at a glance that the modal contribution is 5¢.

Furthermore, the mode also eliminates the extremes in a distribution; that is, it is not affected by the fact that there are a few very small or a few very large items.

For example, in a study of the results obtained in an examination the mode would not be affected, as would the arithmetic mean, by an occasional mark of 0 or one of 100.

Disadvantages of the Mode. One disadvantage of the mode is that in a frequency distribution where the groupings are irregular and no particular class stands out above the others it fails to yield a fair measure of the central tendency of the distribution. As an example of this type, let us consider the accompanying table, which shows the distribution of the weekly wages received by a group of 100 girls in a factory.

Weekly Wage	Number of Workers
$14–$15	3
$15–$16	15
$16–$17	16
$17–$18	12
$18–$19	7
$19–$20	8
$20–$21	5
$21–$22	0
$22–$23	5
$23–$24	0
$24–$25	6
$25–$26	18
$26–$27	5

In this case it is evident that the modal wage is received by the group whose wages are from $25 to $26 a week. An examination of the table, however, reveals that there are many more girls who are working for wages of from $15 to $18 a week; in fact, there are 43 girls in this larger class, while only 29 girls earn from $24 to $27.

If a graph of this table is constructed, it will be found that the curve has several "humps" in it, each of which suggests a possible mode.

If there are two humps in the curve, the case would be classified as a *bimodal distribution*; if there are three humps, it would be a *trimodal distribution*, and so on.

In the above distribution the weighted arithmetic mean, which is easily shown to be $19.89, also seems a little too high as a measure of the central tendency of the entire group. It is therefore evident that this case is one of those in which it is difficult to find a single value that will be a satisfactory measure of the distribution.

In a reasonably symmetric distribution the mode and the arithmetic mean are comparatively close together, and hence the mode may be taken as a rough approximation to the mean.

For example, in a study of the heights of men it was found that the mode was 67 in. and the arithmetic mean was 67.5 in.

Median. If a number of objects are measured with respect to some common characteristic, and the resulting numbers are then arranged in order of size of the measurements, the middle measure of the group is called the *median*. It should be observed that the median cannot be determined until the various items have been arranged in order of size, and it is therefore known as a *position average* or as a *counting average*.

For example, if the ages of a group of seven men are recorded as 24, 57, 30, 53, 32, 40, and 60, these measures of age may be arranged in the order 24, 30, 32, 40, 53, 57, 60. The fourth number from either the right or left is the middle number, and hence the median age in this case is 40 yr.

It is evident that there are three men who are younger and three who are older. That is, a convenient way of thinking of the median is to regard it as the measure which is so chosen that there are just as many measures that are less than the median as there are that are greater.

When there is an even number of items, there is no one middle value. The median is therefore regarded as halfway between the two middle values, or if these two values are the same, either one of them may be taken as the median.

For example, in the case considered on page 133 the marks received by the 8 students were respectively 85 %, 76 %, 91 %, 83 %, 85 %, 81 %, 79 %, and 76 %. Arranged in order of size, we have 76 %, 76 %, 79 %, 81 %, 83 %, 85 %, 85 %, and 91 %. The median in this case may therefore be regarded as halfway between 81 % and 83 %; that is, it is 82 %.

It should be observed that in computing the median, as in computing the mean, all the measures are included; that is, we must count the measures in order to determine the value of the middle one. The median, however, is not affected by extreme values of the variable to as large an extent as the mean.

For example, in a case of statistics relating to the incomes in a large group, the median would be less affected than the mean by ten or twelve incomes of $3,000,000 each, since only the number and not the unusual size of these incomes would affect the result.

Of the three types of averages thus far considered, the mode, which is simply the value of greatest frequency, is the only one that is independent of the number of measures in the group.

Finding the Median. In the frequency table given on page 138 the sum of the numbers in the first four classes is 46. Since there are 7 girls in the next class ($18–$19), it is evident that the median will fall somewhere within this class. We may now assume that the wages of these 7 girls are distributed equally between $18 and $19; that is, that the first girl's wages are between $18 and 18\frac{1}{7}$, the second's between 18\frac{1}{7}$ and 18\frac{2}{7}$, and so on.

This assumption is probably not far from the actual conditions; for example, if the girls were working at piece rates, it would be very nearly correct; and if at weekly wages, the group might be subdivided into those receiving $18, $18.25, $18.50, and $18.75 per week.

The 47th girl in the list is therefore assumed to earn a wage halfway between $18 and 18\frac{1}{7}$, or 18\frac{1}{14}$; the 48th girl a wage of 18\frac{3}{14}$; the 49th, 18\frac{5}{14}$; the 50th, 18\frac{7}{14}$; and the 51st, 18\frac{9}{14}$. The median wage, which is halfway between these last two, is therefore 18\frac{4}{7}$, or $18.57 to the nearest 1¢.

In this distribution we thus have an arithmetic mean of $19.89, a mode of $25.50, and a median of $18.57. Although it is difficult to take any one value as descriptive of this irregular grouping, it is evident that the median wage is the best one to use as a measure of the central tendency.

Disadvantages of the Median. As we have just seen, the median, like the arithmetic mean, has the disadvantage of frequently falling within a class. Furthermore, in cases where there are only a few items the median is affected by extreme values.

For example, the median of $1, $99, and $100 is $99, but obviously there is no single value that will properly describe this distribution.

The principal advantage of the median therefore lies in the ease with which it can be obtained from a table of values that are arranged according to size.

Now that we have discussed the three important measures of central tendency, the arithmetic mean, the mode, and the median, it is evident that it is largely a matter of using common sense in deciding upon which one to use in a given case.

Exercises. Central Tendencies

1. Explain the statement, "The manufacturer of ready-made clothing makes clothes to fit the modal man."

2. What does it mean to say, "The spirit of charity in a community is far more evident in the mode than in the mean"?

3. Find the median number of exercises attempted by the 115 students listed in the table given in Ex. 6 on page 127.

4. Find the mean number of exercises that were attempted by the students in the table of Ex. 6 on page 127.

5. Find the mode for the correct solutions listed in the table in Ex. 6 on page 127.

6. Find the arithmetic mean, the weighted arithmetic mean, the mode, and the median in the table of students' marks given in Ex. 4 on page 126.

7. Find the median length for the 220 ears of corn listed in the table given on page 120.

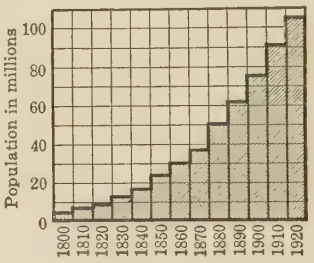
8. In a study of the hereditary tendencies in a group of 60 related individuals, seven points of comparison with the parental type were used. The results of the study were recorded in the following table, in which the first row shows the points of comparison used and the second the number of individuals exhibiting a positive relation to the parental type for this point:

Points of comparison .	0	1	2	3	4	5	6	7
Number of cases . . .	19	5	12	9	4	11	6	3

Find the mean number of points of similarity displayed by the group. Is the tendency of the group toward or away from the parental type? Give the reason for your answer.

9. In the case considered on page 121 do you think that the mode would be a close approximation to the arithmetic mean? Give reasons for your answer, and then verify it by calculating the weighted arithmetic mean from the table.

Geometric Mean. From the accompanying graph it is evident that the population of the United States has been multiplied by approximately 22 in the period from 1800 to 1920. If we take the population as being 5,000,000 in 1800 and 105,000,000 in 1920, the arithmetic mean between these two figures would be 55,000,000. As shown by the graph, however, this estimate of the population for 1860, the midpoint of the period, is far too high; in fact, it is almost double the real figure.



Population of the United States 1800–1920

Population tends to increase more in a geometric series than in an arithmetic series, and hence in such a case we should use the geometric mean (page 99). From the formula $m = \pm\sqrt{al}$, developed in Ex. 3 on page 102, we can show that the geometric mean in this case is 22,910,000. Although somewhat low, this is much nearer than the arithmetic mean.

Similarly, in finding the average annual rate of increase in a city which had a population of 25,000 in 1910 and 50,000 in 1920, we should apply the principles of geometric series.

That is, regarding the population as a geometric series of 11 terms in which $a = 25{,}000$ and $l = 50{,}000$ we have

$$l = ar^{n-1},$$

or $$50{,}000 = 25{,}000 \cdot r^{10}.$$

Then $$r = \sqrt[10]{2},$$

or $$\log r = \tfrac{1}{10} \times 0.3010 = 0.0301;$$

whence $$r = 1.072, \text{ approximately.}$$

That is, the accumulation factor in this case is 1.072, and hence the annual rate of increase is $1.072 - 1$, or 7.2 %, approximately.

For n different values the geometric mean can be found by taking the nth root of the product of the values.

Thus, the geometric mean of 2, 4, 8, 16, and 32 is $\sqrt[5]{2 \cdot 4 \cdot 8 \cdot 16 \cdot 32}$, or 8, while the arithmetic mean is $\frac{1}{5} \times 62$, or 12.4.

Exercises. Averages

1. In a test on equations the numbers of correct solutions given by the 15 members of the class were found to be respectively as follows: 9, 10, 10, 11, 11, 14, 15, 16, 18, 18, 19, 21, 21, 21, 25. Calculate the median, the mode, the arithmetic mean, and the geometric mean. If you were setting a standard for judging the class, which of these four averages should you select? Give the reason for your answer.

2. By the geometric-series method, compute the average rate of yearly increase in a state in which the population increased from 1,751,400 to 2,387,100 during a recent 10-year period. What was the population of the state at the end of 5 yr.? Assuming that this rate will continue, what will be the population 10 yr. after the close of the above period?

In cases where the four-place table of logarithms is not sufficiently accurate, the six-place table on pages 316–333 should be used.

3. The value per farm of all farm property in Minnesota increased during a recent 10-year period from $9456 to $21,221. Assuming the rate of increase to be uniform, find the average annual increase for the period.

4. What per cent of increase per year will double any given population in 20 yr.?

5. In a certain civil-service examination the distribution of points was as follows: (1) education and training, 30; (2) practical experience and fitness, 50; (3) written examination, 20. If an applicant received ratings of 75% on (1), 90% on (2), and 96% on (3), upon what basis should you compute his standing in the examination? What is his standing on this basis?

6. In Ex. 5 compute the standing of an applicant who received ratings of 87% on (1), 92% on (2), and 86% on (3); of 65% on (1), 85% on (2), and 85% on (3).

7. Show why the geometric mean is always smaller than the arithmetic mean.

8. In a study of 300 ears of corn the following frequencies of length, in inches, were recorded, the numbers in the first row corresponding to the midpoints of the various class intervals:

Lengths . . .	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5
Ears	1	1	2	2	4	9	9	11	20

Lengths . . .	8.0	8.5	9.0	9.5	10.0	10.5	11.0	11.5	12.0
Ears	30	38	55	50	35	19	9	4	1

From the above data find the modal length, the median length, and the arithmetic mean length of ear.

9. By means of the formula for l in a geometric series find the mean annual rate of increase in population for the graph shown on page 142. Test the accuracy of your work by finding the population in 1860 and 1890 as determined from this rate and comparing the results with the population in these years as determined from the graph.

10. The following table shows the number of acres planted to each of six different crops, the yield in bushels per acre, and the price per bushel received for each of the six crops:

Crop	Acres	Bushels	Price
Wheat .	20	23	$1.10
Oats . .	13	56	.66
Corn . .	24	35	.95

Crop	Acres	Bushels	Price
Flax . .	12	14	$2.10
Potatoes.	3	175	.50
Barley .	7	25	.90

From the table find the arithmetic mean income per acre for the total acreage. Find which crops returned a smaller income per acre than this average, and find also the per cent by which these crops fell below the average.

11. Is the geometric mean affected more or less by extreme values than the arithmetic mean? Explain your answer.

Exercises. Test on Chapter VI

1. If the cost of each of seven different articles is \$1, \$2, \$2, \$3, \$4, \$5, and \$1000 respectively, what is the mode? What is the median? What is the arithmetic mean? What is the geometric mean? Which, if any, of these results seems to you to give a usable indication of the central tendency in this group?

2. In finding the average rate of travel for five successive hours in which the respective numbers of miles per hour are 12, 15, 18, 21, and 24, which kind of average should you select? What is the reason for your answer? What is the average rate?

3. In a class of 16 students, 2 reported 5 exercises attempted in a certain assignment, 4 reported 6 attempted, 5 reported 7 attempted, 3 reported 8 attempted, and 2 reported 9 attempted. Find the arithmetic mean number attempted.

4. State the advantages and disadvantages of each of the three measures of central tendency, — the arithmetic mean, the mode, and the median.

5. State which one of the three measures of the central tendency of a distribution seems to you to be the best representative measure of the entire distribution. Give the reasons for your answer.

6. State which one of the three measures mentioned in Ex. 4 is least representative, and give the reasons for your answer.

7. Show how two distributions may have the same median and yet differ in some other general aspect.

8. Suppose that a farm-implement company makes 7% on the invested capital in a certain year. This profit is added to the original capital, and 9% is made on the total investment the second year. Proceeding in the same way, the profits are 10% the third year, 12% the fourth year, and 15% the fifth year. Find the average rate of increase for the 5 yr.

In all such cases the student should, as in Ex. 2, state his reason for selecting the particular type of average that he uses.

Exercises. General Review

1. In the formula $q = Av$, which gives the number of cubic feet (q) of water discharged per second through a drain tile, A represents the cross-section area of the tile in square feet, and v the velocity of flow of the water in feet per second. Draw a graph of this formula for a 6-inch tile, and from it find the values of q corresponding to the following values of v: 0.25, 0.50, 1.00, 1.50, 2.00, 2.50, and 3.00.

2. In the formula $q = \sqrt{\frac{gh\pi^2 d^5}{8fL}}$, which gives the number of cubic feet (q) of water discharged per second from a pipe, L is the length and d is the diameter in feet, g is the acceleration due to gravity, h is the hydrostatic head, and f is a friction factor depending upon the kind of pipe and the velocity of flow of the water. From this formula derive one for d.

3. In Ex. 2 find by means of logarithms the value of d in inches when $q = 6$, $g = 32.2$, $f = 0.02$, $L = 3500$, and $h = 40$.

4. Hogs are classified according to weight as follows: heavy, 300 lb.; medium, 250 lb.; light, 200 lb.; and light-light, 150 lb. Find the arithmetic mean price per hundredweight paid for hogs on a day when the number received at a market and the prices per hundredweight for the four classes were as follows:

Class	Heavy	Medium	Light	Light-light
Number	9000	10,000	8000	6000
Price	\$9.70	\$9.50	\$9.25	\$8.40

5. In an arithmetic series in which the first term is one third of the second prove that the sum of the terms is n^2 times the first term.

6. During a 4-year period when the price of corn decreased from \$2.50 to \$1 per bushel, what was the average annual rate of decrease?

CHAPTER VII

INVESTMENTS

Compound Interest. If simple interest on a principal is not paid promptly when it is due, the omitted interest payment does not properly draw interest during the time that it remains unpaid.

In agreements covering financial transactions, penalties are usually provided to assure the prompt payment of the interest.

If the interest is paid promptly, however, it can be invested so as to draw interest. If the rate at which it is invested is the same as that for the original principal, the interest becomes, in effect, an addition to the principal. When the interest is added to the principal at definite intervals, or periods, we have what is known as *compound interest.*

Savings banks allow interest upon interest which is left on deposit after it becomes due; that is, they compound the interest. The whole theory of investments is based upon compound interest; that is, it is assumed that the interest as it comes due is at once reinvested, as is done by the large insurance and trust companies.

The law generally prohibits the making of promissory notes at compound interest, but in case the interest is not paid on a note, a frequent practice is to give a new note for the amount of the unpaid interest, so that the interest is, in effect, added to the principal.

The time elapsing between any two successive dates upon which the interest is added to the principal is called the *conversion period,* or simply the *period.*

For example, if a savings bank adds the interest to the principal on deposit on Jan. 1 and July 1 of each year, the conversion period is 6 mo.

If the period is 1 yr., the interest is said to be *compounded annually*; if 6 mo., it is said to be *compounded semiannually*; and if 3 mo., it is said to be *compounded quarterly.*

Compound Amount. The sum of the principal and the compound interest at a given date is called the *compound amount* at that time. The compound interest can therefore be found by subtracting the original principal from the compound amount.

For example, suppose that on Jan. 1 a man deposits \$200 in a savings bank which pays 4% a year, compounded semiannually. On July 1, 2% of \$200, or \$4, will be added to the principal, and on the following Jan. 1, 2% of \$204, or \$4.08 more, will be added.

Thus, at the end of two periods the compound amount is \$208.08, and the compound interest is \$208.08 − \$200, or \$8.08.

Simple interest for the same period would be only $0.04 \times \$200$, or \$8. Although the difference here is small, over a comparatively long period of time compound interest is of decided advantage to the investor.

Compound Amount of \$1. If we let n represent the number of periods, i the rate of interest for each period, I the interest due for each period, and A the compound amount of a principal of \$1, we may make the following table:

For the first period,	$I = 1 \cdot i = i$
At the end of the first period,	$A = 1 + i$
For the second period,	$I = (1 + i)i$
At the end of the second period,	$A = 1 + i + (1 + i)\, i = (1 + i)^2$
For the third period,	$I = (1 + i)^2 i$
At the end of the third period,	$A = (1 + i)^3$
At the end of the fourth period,	$A = (1 + i)^4$
At the end of n periods,	$A = (1 + i)^n$

Thus, for the compound amount A of \$1 at the rate i for each of n periods, we have the general formula

$$A = (1 + i)^n.$$

For a principal of P dollars the above formula becomes

$$A = P\,(1 + i)^n.$$

In business the factor $(1 + i)^n$, which is the compound amount of \$1, is often called the *accumulation factor*, although the expression "compound amount of 1" is also used.

Exercises. Compound Interest

Find to the nearest 0.00001 the value of

1. 1.03^4. **2.** 1.04^3. **3.** 1.05^5. **4.** 1.045^4. **5.** 1.055^3.

As was seen on page 88, the value of 1.03^4 can be found by expanding $(1 + 0.03)^4$ by the Binomial Formula. Since the result is to be found only to the nearest 0.00001, any terms of the expansion which do not affect the fifth decimal place may be disregarded.

At interest compounded annually, find the amount of

6. \$800 for 3 yr. at $3\frac{1}{2}\%$.

7. \$750 for 2 yr. at 3%.

8. \$925 for 3 yr. at 5%.

9. \$1200 for 4 yr. at $3\frac{3}{4}\%$.

10. \$1550 for 5 yr. at $4\frac{1}{4}\%$.

11. \$1850 for 4 yr. at $6\frac{1}{2}\%$.

12. Since the compound interest is the compound amount less the original principal, show that a general formula for the compound interest I is $I = P[(1 + i)^n - 1]$. Find the value of I when $P = \$800$, $i = 4\%$, and $n = 5$.

At interest compounded annually, find the compound interest on an investment of

13. \$900 for 2 yr. at 4%.

14. \$850 for 3 yr. at $4\frac{1}{2}\%$.

15. \$2400 for 3 yr. at 5%.

16. \$2800 for 4 yr. at $4\frac{1}{4}\%$.

17. A man deposits \$1000 on Jan. 1 in a savings bank which pays interest at $4\frac{1}{2}\%$ a year and compounds the interest on Jan. 1 and July 1 of each year. If the man makes no withdrawals of interest, what is his balance in the bank 3 yr. after making the original deposit?

In this case the number of periods is 6, and the rate per period is $2\frac{1}{4}\%$. The accumulation factor in this case is therefore 1.0225^6.

At interest compounded semiannually, find the compound amount of an investment of

18. \$150 for 2 yr. at 4%.

19. \$100 for $3\frac{1}{2}$ yr. at 5%.

20. \$200 for 3 yr. at $4\frac{1}{2}\%$.

21. \$800 for 2 yr. at 5%.

Logarithms Applied to Interest. In computations involving the formula $A = P(1+i)^n$ logarithms are often convenient. In this work four-place logarithms are not sufficiently accurate, and hence the six-place table on pages 316–333 should be used. It should be remembered, however, that the results are likely to be only approximate in the sixth figure.

For example, find the compound amount of an investment of \$5500 for 6 yr. at 5%, compounded semiannually.

From $A = P(1+i)^n$,

we have $\log A = \log P + n \log (1+i)$

$= \log 5500 + 12 \log 1.025.$

Now $\log 5500 = 3.740363$

$12 \log 1.025 = \underline{0.128688}$

$\log A = 3.869051$

Then $\text{antilog } 3.869051 = 7396.92;$

that is, the required compound amount is thus found to be \$7396.92. As we shall soon learn, the more exact result, found by using a compound-amount table, is 3¢ less than this, or \$7396.89.

Exercises. Logarithms Applied to Interest

At interest compounded semiannually, find to six figures the amount of

1. \$500 for 4 yr. at 4%.
2. \$600 for 5 yr. at 5%.
3. \$850 for 3 yr. at 6%.
4. \$2500 for 5 yr. at $4\frac{1}{2}$%.
5. \$3700 for 6 yr. at $3\frac{1}{2}$%.
6. \$4820 for 5 yr. at $5\frac{3}{4}$%.

7. From the formula $A = P(1+i)^n$ derive a formula for computing the value of n and explain how it can be used.

8. From the formula derived in Ex. 7 find to the next larger integer the number of years that it will take \$2000 to amount to \$3500 at 4%, compounded annually.

9. In his will a man directed that \$50,000, together with the interest thereon, be invested for a period of 20 yr. If the money was invested at $4\frac{1}{2}$%, compounded annually, what was the amount at the end of the 20 yr.?

Nominal and Effective Rates of Interest. As we have seen, when interest is compounded at periods during the year, the interest for the year is greater than in cases where the interest is added to the principal only once a year.

For example, at the rate of 4 % a year, compounded annually, the interest on \$1000 is \$40; at 4 % a year, compounded semiannually, the compound amount is $1.02^2 \times \$1000$, or \$1040.40, and hence the interest is \$40.40.

Interest is usually stated as the rate for borrowing money for 1 yr., which is called the *nominal rate*. The rate per year actually received due to compounding the interest is called the *effective rate*.

In the above case the nominal rate is 4 %, and the effective rate is 4.04 %.

If we let i be the nominal rate, n the number of periods, and e the effective rate, the rate per period is $\frac{i}{n}$, and hence the compound amount of \$1 at the end of 1 yr. is $\left(1+\frac{i}{n}\right)^n$. At the effective rate e, this compound amount is $1+e$, and hence

$$1+e=\left(1+\frac{i}{n}\right)^n,$$

or

$$e=\left(1+\frac{i}{n}\right)^n-1.$$

Table of Compound Amounts. In investment and other work, tables giving the value of the factor $(1+i)^n$ to a large number of decimal places are used. On pages 304 and 305 is given a table showing the value of this factor (the compound amount of \$1) to seven decimal places for ten different values of i and for values of n from 1 to 50.

From this table we can find at once that the compound amount of \$1000 at 4 %, compounded annually, for 5 yr. is \$1216.65. That is, we simply move the decimal point three places to the right in 1.2166529, which is given as the value of $(1+i)^n$ when $i=0.04$ and $n=5$.

Similarly, at 5 %, compounded semiannually, the amount of \$2000 for 10 yr. is equal to the amount of \$2000, compounded annually for 20 yr. at $2\frac{1}{2}$ %. From the table, $(1+i)^n=1.6386164$ when $i=0.025$ and $n=20$, and hence the required amount is $2000 \times \$1.6386164$, or \$3277.23.

Exercises. Rates and Amounts

1. By means of the formula on page 151 find the effective rate of interest on an investment at 5%, compounded quarterly.

In using logarithms it is more convenient to write the effective-rate formula as $\log (1 + e) = n \log \left(1 + \frac{i}{n}\right)$, and to find the value of $1 + e$ first.

2. From the table on page 304 verify the result of Ex. 1 by finding the compound amount of \$1 at $1\frac{1}{4}\%$ for four periods.

Find the effective rate of interest for each of the following nominal rates, compounded as specified:

3. 8%, semiannually.

4. 7%, semiannually.

5. 6%, quarterly.

6. 6%, monthly.

7. $5\frac{1}{2}\%$, semiannually.

8. $5\frac{3}{4}\%$, quarterly.

9. $4\frac{1}{2}\%$, semiannually.

10. $4\frac{3}{4}\%$, quarterly.

11. The compound-amount table shows that the amount of \$1 at 6% is \$1.90 when $n = 11$ and \$2.01 when $n = 12$; that is, a principal of \$1 doubles itself in a little less than 12 yr. at 6%, compounded annually. From the table find the approximate time in which an investment will double itself at each of the other rates, compounded annually, given in the table.

12. From the compound-amount formula $A = P(1 + i)^n$ derive a formula for $\log (1 + i)$, and from it find the approximate rate of interest, compounded annually, at which \$1000 must be invested in order to double itself in 10 yr.

13. At what nominal rate of interest, compounded semiannually, must \$2500 be invested in order to amount to \$3500 at the end of 8 yr.? What effective rate must be received?

14. A man leaves \$40,000 to a college on condition that the interest shall be reinvested until the original gift has increased to \$60,000, at which time the whole amount is to be used for a swimming pool. If the money is invested at 5%, compounded annually, how many years must elapse before it can be used?

Problems in Compound Interest. The formula for the compound amount of an investment,

$$A = P(1 + i)^n,$$

involves the four letters A, P, i, and n. If the values of any three letters are known, we can find the fourth. That is, we have four problems in compound interest, as follows:

1. Find the value of A when P, i, and n are known; that is, find the compound amount of a given principal invested at a fixed rate of interest for a certain period of time.

We have already learned how to solve this problem in the following ways: (1) by finding the value of $(1 + i)^n$ by the Binomial Formula; (2) by using six-place logarithms; and (3) by using a compound-amount table.

Since we can find the compound interest by simply subtracting the principal from the compound amount, finding the interest is obviously only a special case of this problem.

2. Find the value of n when P, A, and i are known; that is, find the time in which a certain principal will accumulate to a specified amount at a given rate of interest.

This problem has already been considered in a few exercises, particularly in relation to the time that it takes a sum of money to double itself.

3. Find the value of i when P, A, and n are known; that is, find the rate at which a given sum of money must be invested in order to accumulate to a specified amount in a certain time.

This problem also has been briefly considered. Since an investor would hardly put his money into an investment paying the desired rate if he had any doubts about its safety, it is evident that this problem is of less practical importance than the other two.

4. Find the value of P when i, A, and n are known; that is, find the sum of money which must be invested at a certain rate in order to accumulate to a desired amount within a given time.

Next to the first problem, this case is the one which occurs most frequently in the business world. It appears in calculations relating to the settlement of certain types of long-term financial obligations and in those relating to the determination of the cash value of income-producing property or contracts of various kinds.

Present Value. When a debt is due at some future date, the sum that must be invested today at the current rate of interest in order to accumulate to the amount of the debt on the date it is due is called the *present value* of the debt.

That is, the present value is the value of P in the formula

$$A = P\,(1+i)^n;$$

whence

$$P = A\frac{1}{(1+i)^n}.$$

The table on pages 306 and 307 gives to seven decimal places the value of $\frac{1}{(1+i)^n}$, and hence P can be easily found.

For example, suppose that a business man owes \$10,000 due in 5 yr., and that at the end of the second year he desires to pay the debt in full. If, at that time, money can be invested so as to earn $4\frac{1}{2}$ %, compounded annually, in finding the present value we have $A = 10{,}000$, $i = 0.045$, and $n = 3$. From the table on page 307 we find that $\frac{1}{(1+i)^n} = 0.8762966$ when $i = 0.045$ and $n = 3$, and hence

$$P = 10{,}000 \times 0.8762966 = 8762.966.$$

That is, if the man pays the lender \$8762.97, the latter can then invest it so that it will amount to \$10,000 at the end of 3 yr. The lender should be willing to agree to this since he obtains his money 3 yr. before it is due. It is assumed, as stated before, that the interest as it comes due is reinvested, thus having the effect of compound interest.

Present Value of a Debt Bearing Interest. If we let F represent the face of a debt bearing interest at the rate r, which is different from the current rate i, the amount A of the debt at the end of n years is given by the formula

$$A = F(1+r)^n.$$

Substituting this value of A in the above formula for P, we have

$$P = F(1+r)^n\frac{1}{(1+i)^n}.$$

For example, if a debt of \$800, which bears interest at 5 %, is due in 4 yr., and the current rate of interest is 4 %, we have

$$P = 800 \times 1.2155063 \times 0.8548042 = 831.215\cdots;$$

that is, the present value under these conditions is \$831.22.

Exercises. Present Value

1. When the current rate of interest on investments is 5%, what is the present value of $2000 due in 5 yr.?

2. When the current rate of interest is $4\frac{1}{2}$%, what is the present value of a mortgage for $12,500 due in 10 yr. and bearing interest at 6%?

In cases like this where the multiplication is long, the student may, unless otherwise directed, use six-place logarithms. The results by this method will generally be accurate within a cent or two, which is sufficient for our present purposes.

3. If money is worth 4%, compounded annually, what is the present value of $4200 due in 12 yr.?

An expression like "money is worth 4%" is frequently used in expressing the current rate of interest at which investments can be made.

4. If money is worth 5%, compounded semiannually, what is the present value of a debt for $6500 which bears interest at 4% a year and is due in 5 yr.?

5. A man buys a house for $6000 and agrees to pay $1000 down and $1000 at the end of each year for 5 yr., with interest at 6% on the unpaid balance. If the current rate of interest is 5%, compounded semiannually, what sum of money will liquidate the debt at once?

The required result is obviously the sum of the cash payment and the present values of five debts of $1000 each, bearing interest at 6% and due in 1 yr., 2 yr., 3 yr., 4 yr., and 5 yr. respectively.

6. At the birth of his son a father plans to set aside a sum that shall amount to $10,000 on the day the boy is 21 yr. old. If he can invest the sum at 4%, compounded semiannually, how much must be set aside?

7. A man wishes to divide $30,000 between his two daughters, who are respectively 16 yr. and 19 yr. old, in such a way that as each girl reaches the age of 21 the amount received will be equal. If the money is invested at 5%, compounded annually, how much should he set aside for each daughter?

Ordinary Annuity. A series of equal periodic payments in which the first payment is made at the end of the first period is called an *ordinary annuity.*

Although the name "annuity" suggests annual payments, the installments are frequently paid semiannually, quarterly, or at other periods.

The sum of all the payments, together with the interest on each one compounded at the end of each period, is called the *final value* of the annuity, and is usually represented by $A_{\overline{n}|}$.

If we let n represent the number of payments of R dollars each and i the rate of interest per period, the compound amount A_1 of the first payment, which draws interest for $n - 1$ periods, is given by the formula

$$A_1 = R(1 + i)^{n-1}.$$

Similarly, $A_2 = R(1 + i)^{n-2}$, $A_3 = R(1 + i)^{n-3}$, $\cdots$, $A_n = R$, and hence for the final value $A_{\overline{n}|}$ we have

$$A_{\overline{n}|} = R(1 + i)^{n-1} + R(1 + i)^{n-2} + \cdots + R(1 + i) + R,$$

or, removing the factor R and reversing the order, we have

$$A_{\overline{n}|} = R[1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-1}].$$

Since the expression in brackets is a geometric series of n terms with the first term 1 and the ratio $1 + i$, we have

$$A_{\overline{n}|} = R\frac{1 \cdot (1 + i)^n - 1}{(1 + i) - 1},$$

or

$$A_{\overline{n}|} = R\frac{(1+i)^n - 1}{i}.$$

Values of the fraction $\frac{(1 + i)^n - 1}{i}$, which is usually called the *amount of $1 per annum,* are given in the table on pages 308 and 309 for different values of i and n.

For example, when the current rate of interest is 5 %, compounded semiannually, for an annuity of $200, paid every 6 mo. for 3 yr., we have

$$A_{\overline{n}|} = 200\left[\frac{(1 + 0.025)^6 - 1}{0.025}\right] = 200 \times 6.3877367 = 1277.54734;$$

that is, the final value of the annuity is thus found to be $1277.55.

Present Value of an Ordinary Annuity. The sum of the present values of all the payments of an ordinary annuity is called the *present value* of the annuity. This present value is evidently the sum which must be invested at the current rate of interest in order to accumulate to an amount equal to the final value of the annuity on the date it is due.

If we let $P_{\overline{n}|}$ represent the present value of an annuity of n payments of R dollars each, and i the interest rate per period, it is evident from the definition just given that

$$P_{\overline{n}|}(1+i)^n = A_{\overline{n}|};$$

whence $$P_{\overline{n}|}(1+i)^n = R\frac{(1+i)^n - 1}{i},$$

or $$P_{\overline{n}|} = R \cdot \frac{1}{i}\left[1 - \frac{1}{(1+i)^n}\right].$$

The table on pages 310 and 311 gives to seven decimal places the values of the factor $\frac{1}{i}\left[1 - \frac{1}{(1+i)^n}\right]$ for different values of i and n.

In taking the expression $P_{\overline{n}|}(1+i)^n$ to represent the compound amount of the present value in the above discussion, we are applying to $P_{\overline{n}|}$ the same rate of interest and the same period as to the payments R. As we shall see later (page 161), if the rates and periods are different, the formula for the present value is changed.

Amortization. The process of paying off or of liquidating a financial obligation is called *amortization*. For example, if a man borrows \$5000 to be repaid with interest at 5% in ten equal annual installments, the debt is said to be *amortized* by this plan of payment.

In this case the problem is to find the size of the annual payment necessary to extinguish the debt. As we shall see later, an annual payment of \$647.52 for 10 yr. will be necessary.

Building and Loan Associations frequently lend money according to a plan which requires a fixed monthly payment for interest and a fixed monthly payment to reduce the principal. As the principal is paid off, the excess in the interest payments is used to reduce the time during which the borrower has to make the payments.

Sinking Fund. When the face of a debt is not reduced by regular payments (that is, when only the interest as it becomes due is paid by the borrower), it is often customary to invest periodically a fixed amount so as to accumulate funds for the payment of the principal when it becomes due. The amount of this periodic investment is determined in such a way that the sum of all the investments, plus the compound interest on each one, equals the face of the debt at maturity. The annuity thus created is known as a *sinking fund.*

For example, a manufacturing company might borrow \$200,000 by issuing bonds due in 20 yr. and bearing interest at 6 %. Besides paying the interest to the bondholders, the company might set aside each year, investing it at the current rate of interest, an amount that will be sufficient to accumulate to \$200,000 at the end of the 20 yr.

The most important problems in connection with sinking funds are as follows:

1. To find R when n, i, and $A_{\overline{n}|}$ are known; that is, to find the size of the n periodic investments at the interest rate i which it is necessary to make in order that the final amount $A_{\overline{n}|}$ shall equal the face of the debt.

Since the amounts set aside periodically in a sinking fund form an annuity, the final value of which must equal the face of the debt, this problem evidently requires the value of R in the formula for $A_{\overline{n}|}$ on page 156. In the above case, it can be shown that at 4 %, compounded annually, an annual investment of \$6716.35 will accumulate to \$200,000 in 20 yr.

2. To find n when R, i, and $A_{\overline{n}|}$ are known; that is, to find the number of investments of R dollars each that must be made at the rate i in order to accumulate an amount equal to the face $A_{\overline{n}|}$ of the debt.

Since the time (the number of payments) in which a debt must be paid off is generally determined by other considerations, this problem is of less practical importance than the first one.

3. To find the sum due after the kth payment.

Such a problem arises when it is desirable to discharge the remainder of the obligation in one payment.

Sum Due after k Payments. After k of the n installments have been paid into a sinking fund, it is evident that the balance which remains unpaid is the present value of an annuity of $n - k$ payments. If we let S_k represent the sum due after the kth payment, we therefore have

$$S_k = R \cdot \frac{1}{i}\left[1 - \frac{1}{(1+i)^{n-k}}\right].$$

The details of the amortization of an indebtedness are frequently put in tabular form showing the interest payments and the amounts applied on the principal. Such a table is called an *amortization schedule.*

Annuity Due. An annuity in which the first payment is made at the beginning instead of the end of the first period is called an *annuity due.*

Since life insurance premiums are paid in advance, the premium payments over a number of years form such an annuity.

In the case of an annuity due, the compound amounts of the successive payments are given by

$$A_1 = R(1+i)^n, \quad A_2 = R(1+i)^{n-1}, \ldots, \quad A_n = R(1+i),$$

and hence for the final value of an annuity due, which we may represent by $a_{\overline{n}|}$, we have

$$a_{\overline{n}|} = R(1+i)^n + R(1+i)^{n-1} + \ldots + R(1+i)^2 + R(1+i).$$

If we remove the factor $1 + i$, this series will be the same as the one on page 156, and hence

$$a_{\overline{n}|} = R(1+i)\frac{(1+i)^n - 1}{i}.$$

Similarly, the present value of such an annuity is the sum of the first payment R and the present value of an annuity of $n - 1$ payments; that is, for the present value $p_{\overline{n}|}$ we have

$$p_{\overline{n}|} = R + P_{\overline{n-1}|},$$

where $P_{\overline{n-1}|}$ is the present value at the rate of interest i of an ordinary annuity of $n - 1$ payments of R dollars each.

The value of $P_{\overline{n-1}|}$ is easily found from the formula for $P_{\overline{n}|}$ on page 157.

Exercises. Annuities

1. When money is worth 4%, compounded semiannually, what is the final value of an annuity of $50 payable at the end of each 6 mo. for 10 yr.?

2. When money is worth 5%, compounded semiannually, what is the present value of a contract which specifies payments of $600 at the end of each 6 mo. for 20 yr.?

3. If 5%, compounded quarterly, is obtainable at current rates, what is the present value of an annuity of $1000 a year, payable at the end of each 3 mo., for 10 yr.?

The six-place table of logarithms may be used whenever convenient in saving labor in long computations, or in cases where it is necessary to take values of i or n not given in the investment tables.

4. A man deposits $50 every 3 mo. for his son, beginning on the day the child is born. If interest on the account is compounded quarterly at 5% per annum, how much will the fund amount to on the day the son reaches the age of 12, the last payment having been made 3 mo. before this date?

5. By the terms of his father's will, a son is to receive the rental, which amounts to $6000 a year, from a building in which the rents are collected monthly in advance. At 6%, compounded monthly, what sum paid to the son on Jan. 1 of each year would equal the amount of the bequest?

6. After agreeing to pay down $1500 and to pay $7500 in five equal semiannual installments without interest, a man is offered the alternative of settling the whole debt by a single cash payment of $8500. If 5% per annum, compounded semiannually, is obtainable at current rates, which is the better plan for the man to follow? How much better for him is it?

7. At 4%, compounded annually, find the accumulated value of the premiums paid on a 20-payment life-insurance policy for $10,000 on which the annual premium rate is $32.12 per $1000, payable in advance.

Payment Intervals of Two or More Years. When the payments in an ordinary annuity are made at intervals of t years, instead of annually, the ratio in the series

$$A_{\overline{n}|} = R[1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-1}],$$

which was given on page 156, becomes $(1 + i)^t$ instead of $1 + i$. The formulas for $A_{\overline{n}|}$ and $P_{\overline{n}|}$ for such an annuity are therefore

$$A_{\overline{n}|} = R\frac{(1+i)^{nt}-1}{(1+i)^t-1},$$

and

$$P_{\overline{n}|} = R\frac{(1+i)^{nt}-1}{[(1+i)^t-1](1+i)^{nt}}.$$

A modification of the above case frequently occurs in commercial work, for example, in lumbering, where the first payment (or increment of profit) occurs at the end of x years, where $x < t$. The effect of this is to increase the value of the property for the total period by an amount that is equal to the earnings for $t - x$ years; that is, the owner does not have to wait until the end of t years for his first return.

The effect is the same as if the total earning period were prolonged $t - x$ years; that is, we have

$$A_{\overline{n}|} = R\frac{[(1+i)^{nt}-1](1+i)^{t-x}}{(1+i)^t-1},$$

and

$$P_{\overline{n}|} = R\frac{[(1+i)^{nt}-1](1+i)^{t-x}}{[(1+i)^t-1](1+i)^{nt}}.$$

For example, suppose that a man owns timberland which will yield ten cuts valued at \$150,000 each, the first cut to come in 16 yr. and the others at intervals of 25 yr. In finding the present value of the cuts at 3 %, compounded annually, we therefore have

$$P_{\overline{n}|} = 150{,}000\frac{(1.03^{250}-1)\,1.03^{9}}{(1.03^{25}-1)\,1.03^{250}}.$$

By six-place logarithms we can find that $P_{\overline{n}|} = \$178{,}829$, approximately.

If $x > t$, the factor $(1 + i)^{t-x}$ will have a negative exponent, and will therefore appear in the denominator. The effect will be to decrease rather than extend the earning period.

Exercises. Annuities

1. A contract calls for payments of \$500 every 3 yr. for 24 yr., the first payment to be made 3 yr. after the date of the contract. At 5%, compounded annually, find the final value and also the cash (present) value of the contract.

2. If money is worth 3%, compounded annually, what is the final value, and also the cash value, of an agreement specifying 10 payments of \$60,000 each every 30 yr., the first payment to be made in 12 yr. from the date of the agreement?

3. If the first payment in Ex. 2 is not to be made until 40 yr. from the date of the agreement, what will be the final value? the present value?

4. For a piece of forest property which yields a gross return of \$150,000 every 25 yr. the annual expenses for taxes, protection, and supervision are estimated at \$600. If money is worth $3\frac{1}{2}$%, compounded annually, what should be the total net return from the property for a period of 75 yr.?

The item of \$600 for annual expenses may be considered as paid at the end of each year. If this amount did not have to be paid for the purposes mentioned, it could be invested at the prevailing rate, and hence the expenses should be compounded annually at $3\frac{1}{2}$%.

5. A piece of property is sold under a contract calling for a payment of \$3600 every 3 yr. for 21 yr. If money is worth 4%, compounded annually, what is the equivalent cash value of the property?

6. An appraiser wishes to estimate the final value and the present value of a piece of forest land which will yield a crop worth \$135,000 every 25 yr. Estimating 10 cuts in all, the first cut to be made in 15 yr., what final value and what present value should he take on the basis of a current rate of $3\frac{1}{2}$%, compounded annually?

7. In Ex. 6, what will be the results if the first cut is not made until the end of 40 yr.?

Perpetuities and Capitalization. An annuity in which the payments are to continue forever is called a *perpetuity*; that is, as n increases indefinitely, an annuity becomes a perpetuity. In such a case the final value $A_{\overline{n}|}$ ceases to have any practical significance. The payments, however, may be considered as the earnings on an investment that is equal to the present value $P_{\overline{n}|}$ of the perpetuity; that is, the present value is the *capital* upon which the earnings are based.

In the case of a perpetuity in which the payments are made annually, it is evident that as $n \to \infty$, $\frac{1}{(1+i)^n} \to 0$, and hence the formula

$$P_{\overline{n}|} = R \cdot \frac{1}{i}\left[1 - \frac{1}{(1+i)^n}\right]$$

may be written as $$P_{\overline{n}|} = \frac{R}{i}.$$

For example, when money is worth 6 %, a man who is earning a yearly salary of \$4800 may regard his time as an investment valued at \$4800 ÷ 0.06, or \$80,000; that is, he may regard his time as capitalized at \$80,000.

In case the payment interval of a perpetuity is t years, where $t \geqq 2$, the first formula for $P_{\overline{n}|}$ on page 161 becomes

$$P_{\overline{n}|} = R \frac{1}{(1+i)^t - 1}.$$

For example, in the dry-farming belt a crop is raised only every second year, the ground lying fallow during the off year. In computing at 5 %, compounded annually, the value as an investment of a piece of land from which a farmer obtains every second year a wheat crop valued at \$2000, we have

$$P_{\overline{n}|} = 2000 \cdot \frac{1}{1.05^2 - 1} = 19{,}512.2.$$

That is, to the nearest dollar the value of the land is \$19,512.

Similarly, when the first payment interval x of a perpetuity is different from the regular interval t, as in the second formula for $P_{\overline{n}|}$ on page 161, we have

$$P_{\overline{n}|} = R \frac{(1+i)^{t-x}}{(1+i)^t - 1}.$$

Exercises. Perpetuities and Capitalization

1. If money is worth 5%, what is the capitalized value of the time of a man who earns $2400 a year?

2. When money is worth 6%, what is the capitalized value of a farm from which a yearly income of $3000 is derived?

3. An estate valued at $350,000 yields an annual income of $20,000 to the heirs. If this income continues indefinitely, what per cent does the estate yield?

4. Allowing 3% a year for depreciation, what is a reasonable cash price for a house which rents for $75 a month when money is worth 6%?

5. When 6% is being paid for money for building homes, and depreciation is computed at 3% a year, what monthly rental will cover these items for a house which costs $6000?

6. When money is worth $5\frac{1}{2}$%, what is a reasonable capitalized value to take for a manufacturing business in which the yearly profits average $16,500?

7. The annual total rental received from an office building is $100,000, and the annual depreciation is computed at 3% of the value of the building. If money is worth $5\frac{1}{2}$%, what is the capitalized value of the building?

8. When money is worth 5%, what is the cash (present) value of an estate that will return indefinitely to the heirs a regular income of $50,000 every 12 yr.

That is, the payment interval in this case is 12 yr.

9. It is estimated that a piece of forest property will yield indefinitely a return of $150,000 every 30 yr., the first return to come in 12 yr. If money is worth 4%, what is the cash value of the property?

10. Solve Ex. 9, supposing that the regular payment interval is 30 yr., as before, but that the first return from the property comes in 40 yr. instead of 12 yr.

Exercises. Test on Chapter VII

1. What is the effect upon the total interest received from an investment at simple interest when the time is doubled? when the time is cut in half?

2. Consider Ex. 1 when the investment is made in such a way that the interest is compounded annually.

3. If a man invests \$200 each year for 20 yr. at 4%, compounded semiannually, what will be the total amount of his investment at the end of the 20 yr.?

4. If the man mentioned in Ex. 3 stops making the annual payments of \$200 at the end of the 20 yr. and merely reinvests the interest as it comes due, what will be the amount of his investment at the end of 10 yr. more?

5. At 5%, compounded semiannually on the balance at the beginning of each period, how much will \$25 a month amount to in 20 yr. if paid into a building and loan association?

6. If money is worth 5%, compounded semiannually, how much must a farmer set aside at the end of each year in order to provide funds for the replacement of a grain binder which costs \$225 and has an estimated life of 12 yr.?

7. A contract calls for payments of \$1200 every 4 yr. for 32 yr., the first payment to be made 4 yr. after the date of the contract. At 4%, compounded annually, what is the cash value of the contract?

8. A man signs an agreement to purchase a piece of timberland by paying \$5000 down and \$5000 every 5 yr. for 25 yr. If money is worth $4\frac{1}{2}$%, compounded annually, what is the cash value of the property?

9. If a city borrows \$150,000 at $4\frac{1}{2}$% and can obtain 4%, compounded annually, on its investments, how large an annual appropriation must be made to pay the interest and to provide for a sinking fund that will cancel the debt in 20 yr.

Exercises. General Review

Solve each of the following equations:

1. $x^2 + 5x + 6 = 0$.
2. $2x^2 - 5x - 3 = 0$.
3. $4.7x^2 - 0.9x = 0$.
4. $\sqrt{x-7} - 4 = 0$.
5. $0.32x^2 + 1.73x = 0$.
6. $6x^2 - 5x - 6 = 0$.
7. $3y^2 + 11y - 4 = 0$.
8. $\sqrt{x-11} = \sqrt{x} - \sqrt{11}$.

9. From the law of falling bodies, $d = v_0 t + \frac{1}{2} gt^2$, derive a formula for t.

10. Draw a bar graph to show the temperature variation in a northern city in which the noon temperatures on ten successive winter days were as follows: 42°, 31°, 4°, − 11°, − 7°, − 18°, 0°, 12°, 50°, 42°. Why is a bar graph the preferred method of representing these statistics?

11. The following table shows the grades obtained in an examination which was taken by a class containing 220 students. The results are grouped in ten classes, the 10% class including all whose grades were more than 0 and less than 15%, the 20% class all whose grades were from 15% to 25%, and so on:

Class	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%
Students	5	8	17	24	41	53	37	22	11	2

Plot the frequency curve showing the distribution, and find the arithmetic mean mark, the modal mark, and the median mark for the entire group.

12. A farmer decides to buy a farm for $16,000, paying for it in equal installments at the end of each year for 25 yr. If money is worth 5%, compounded annually, what will be the size of each installment?

13. If a farmer has machinery which 10 yr. hence it will cost $2000 to replace, what annual amount should he set aside as a sinking fund, on which interest is compounded annually at 4%, in order to provide for the replacement of the machinery?

CHAPTER VIII

DAIRY PROBLEMS

Nature of the Work. When we speak of $3\frac{1}{2}\%$ milk, we mean milk that contains $3\frac{1}{2}\%$, by weight, of butter fat. One of the most common problems in dairying relates to grading milk or cream according to the per cent of butter-fat content.

Another important problem relates to mixing milk or cream of different grades in order to obtain a specified grade.

Various modifications of these problems occur when it is necessary to consider the percentages of total solids or of non-fat solids and specific gravities (see Table 9 on page 349).

Formula 1. It is easy to develop convenient formulas for solving these types of problems. For example, suppose that it is desired to find how many pounds of m per cent milk and c per cent cream can be obtained from a given quantity (Q pounds) of milk testing q per cent.

If we let M represent the number of pounds of m per cent milk and C the number of pounds of c per cent cream, we have

$$Q = M + C \tag{1}$$

and

$$qQ = mM + cC. \tag{2}$$

From (1),

$$cQ = cM + cC. \tag{3}$$

Subtracting equation (2) from equation (3), we have

$$(c - q)Q = (c - m)M;$$

whence

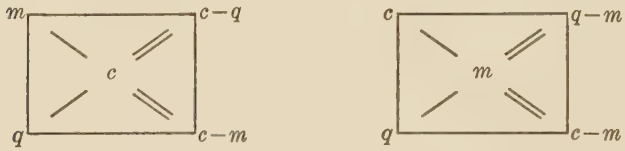
$$M = \frac{c - q}{c - m} Q.$$

Similarly, by eliminating M in (1) and (2), we have

$$C = \frac{q - m}{c - m} Q.$$

Dairyman's Rectangle. A convenient method of remembering and applying the formulas on page 167 involves the use of the *dairyman's rectangle,* which is shown in the two figures below.

$$M = \frac{c-q}{c-m}\,Q \qquad C = \frac{q-m}{c-m}\,Q$$



In solving problems by means of these rectangles we proceed according to the following rule:

Place the per cent for the given quantity of base milk at the lower left corner, the per cent for the required amount of milk (or cream) at the upper left corner, and the per cent for the required amount of cream (or milk) at the center. Subtract diagonally as shown in the diagrams, regarding each result as positive, and multiply the ratio of the results by the weight of the base milk.

For example, how many pounds of $3\frac{1}{2}\%$ milk and 20% cream will 500 lb. of $5\frac{1}{2}\%$ milk yield?

Using the rectangles as stated above, we may proceed as follows:

$$M = \frac{c-q}{c-m}\,Q \qquad C = \frac{q-m}{c-m}\,Q$$

Hence $M = \dfrac{14\frac{1}{2}}{16\frac{1}{2}} \cdot 500$

$= \frac{2}{3}\frac{9}{3} \cdot 500$

$= 439.39 \cdots.$

Hence $C = \dfrac{2}{16\frac{1}{2}} \cdot 500$

$= \frac{4}{3}\frac{}{3} \cdot 500$

$= 60.60 \cdots.$

That is, from the $5\frac{1}{2}\%$ milk it will be possible to obtain 439.4 lb. of $3\frac{1}{2}\%$ milk and 60.6 lb. of 20% cream.

Check. 439.4 lb. + 60.6 lb. = 500 lb.

Exercises. Formula 1

1. How many pounds of 3% milk and 21% cream will 1400 lb. of 5% milk yield?

2. How much skim milk and 21% cream can be obtained from 1200 lb. of $3\frac{1}{2}$% milk?

Although skim milk may contain a slight percentage of butter fat, in such cases the amount of fat is entirely disregarded.

3. What was the per cent of butter fat in a quantity of base milk from which 880 lb. of $3\frac{1}{2}$% milk and 120 lb. of 20% cream was obtained?

4. If 2600 lb. of $4\frac{1}{2}$% milk yields 2370 lb. of 3% milk, and the balance is cream, what is the weight of the cream? What is the per cent of butter fat in the cream?

5. If a dairy farm produces 40,000 lb. of 4.2% milk during the year, what is the income if the milk is sold straight (that is, without being standardized) at $3.50 per 100 lb.? What is the income if the output is standardized and sold as 3.6% milk and 20% cream, the milk selling for $3.00 per 100 lb. and the cream for $1.25 per gallon.

For the weights per gallon of milk and cream of different grades the table on page 349 should be consulted.

6. If 10 Jersey cows produce an average of 5600 lb. each of $5\frac{1}{2}$% milk per year, what is the difference in gross income between selling the milk straight at $4.80 per 100 lb. or standardizing it and selling it at $3.00 per 100 lb. for 3.6% milk and $1.35 per gallon for 21% cream?

7. A farmer has 20 cows which produce an average of 22 lb. each of 3.6% milk per day. Which of the following methods of selling the daily output will yield the greatest gross income to the farmer: (1) as straight milk at $3.00 per 100 lb.; (2) as 25% cream at $1.50 per gallon and skim milk at 35¢ per 100 lb.; (3) as butter fat in sour cream at 70¢ per pound of butter fat and skim milk at 35¢ per 100 lb.?

8. A dairy farmer's records showed that for a herd of 10 cows the number of pounds of milk produced by each cow during the year and the butter-fat content of the milk were as follows:

Cow	1	2	3	4	5	6	7	8	9	10
Milk	7200	6100	5150	2900	6200	4100	4000	10,000	4700	3400
Fat	3.6%	3.8%	4.0%	3.0%	4.2%	2.4%	6.0%	3.6%	5.0%	2.6%

Allowing 20% extra for "overrun," find the number of pounds of butter that can be made from the milk.

The "overrun" is the excess in weight of the finished butter which is due to the water left after washing the butter and to the salt added to the finished product.

9. How much will the farmer mentioned in Ex. 8 receive for his butter at 60¢ per pound?

10. In Exs. 8 and 9, at what price per pound should the farmer compute the value of the pure butter fat?

11. In Ex. 8 suppose that it costs $100 per cow per year to care for the cows numbered 4, 6, and 10, and $110 per cow per year for each of the others. If the skim milk is worth 30¢ per 100 lb. and the value of the manure averages $10 per cow per year, how much is the farmer's profit?

12. In Exs. 8–11, if the farmer had disposed of the cows numbered 4, 6, and 10 at the beginning of the year, how much more profit would he have received? By what per cent would his profit have been increased?

13. In Ex. 8 determine the mean annual production per cow and the mean butter-fat content.

14. In Ex. 13, if the farmer had disposed of the cows as mentioned in Ex. 12, what would have been the mean annual production per cow and the mean butter-fat content? By what per cents would these averages be increased over those determined in Ex. 13?

Formula 2. Another important problem is to find how many pounds of a certain grade of milk or cream must be mixed with a given quantity of milk (or cream) to produce a desired grade of milk (or cream).

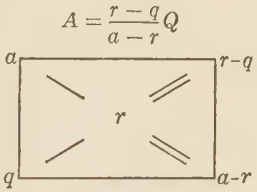
In developing a formula for this problem, we may let A represent the number of pounds of a per cent milk or cream which must be added to the given quantity (Q pounds) of q per cent milk (or cream) in order to produce the required milk (or cream) containing r per cent butter fat. We then have

$$qQ + aA = r(Q + A)$$
$$= rQ + rA;$$

whence $$A(a - r) = Q(r - q),$$

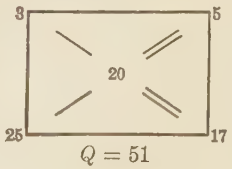
or $$A = \frac{r - q}{a - r} Q.$$

In using the dairyman's rectangle shown below, we first place the per cent for the required mixture in the center, the per cent for the given quantity at the lower left corner, as before, and the per cent for the added milk (or cream) at the upper left. We then subtract diagonally, considering both results as positive, and multiply the ratio of these results by the weight of the given quantity of milk (or cream).



For example, how many pounds of 3% milk must be added to 51 lb. of 25% cream in order to produce cream which will test 20% fat?

Writing the given per cents in their proper places in the dairyman's rectangle, as shown in the figure at the right, we have



$$A = \tfrac{5}{17} \cdot 51$$
$$= 15.$$

That is, it will be necessary to add 15 lb. of the 3% milk in order to obtain the 20% cream, as required. The total amount of 20% cream will therefore be 51 lb. + 15 lb., or 66 lb.

Exercises. Formula 2

1. In order to produce 4% standardized milk, what proportions of 3.7% milk and 20% cream must be used?

2. What proportions of 5.6% milk and skim milk are necessary in order to produce 3.5% milk?

3. What proportions of 45% cream and 3.6% milk are necessary in order to produce 21% cream?

4. How many pounds of 3.6% milk are required in order to reduce 57 lb. of 28% cream to 20% cream? How much 20% cream will there be?

5. If a dairyman has 50 gal. of cream that tests 30% fat, how many gallons of 5.6% milk will have to be added in order to produce 20% cream?

6. How many pounds of 20% cream must be mixed with 250 lb. of $2\frac{1}{2}$% milk in order to produce $3\frac{1}{2}$% milk?

7. How many pounds of 4% milk must be mixed with 100 lb. of 28% cream in order to produce 20% cream?

8. How many pounds of skim milk must be added to a mixture of 50 lb. of 30% cream and 400 lb. of 4% milk in order to produce $3\frac{1}{2}$% milk?

9. What proportions of skim milk and 20% cream are needed in order to produce $3\frac{1}{2}$% milk? How many pounds of each are needed in order to produce 500 lb. of the $3\frac{1}{2}$% milk?

10. If, when 200 lb. of milk is mixed with 10 lb. of 30% cream, the result is $3\frac{1}{2}$% milk, what was the per cent of butter fat in the original milk?

11. If a farmer wishes to obtain 3000 lb. of 3.6% milk, how many pounds of skim milk and 20% cream will he need to use?

12. A farmer has 6 Holstein cows, each of which produces 8000 lb. of 3% milk per year. How many Jersey cows, each producing 6000 lb. of $4\frac{1}{2}$% milk per year, should he buy in order that the mixed milk shall test $3\frac{1}{2}$%?

Conversion Formulas for Powdered Milk. In order to obtain satisfactory results in precipitating the powder in the manufacture of powdered milk, and to meet legal requirements for the finished product, it is necessary to convert the natural milk in such a way that there is a definite ratio between the amounts of butter fat and of total solids in the residue milk.

In developing formulas that can be applied to any natural milk, we may let Q represent the quantity (in pounds) of natural milk testing q per cent fat and containing s per cent of total solids used in making the powdered milk. It is then necessary to determine the amount of 20% cream (containing 27.13% total solids) which must be removed in order that the fat content of the residue milk shall be 27% of the total solids. The per cent of fat in the residue milk is then easily determined.

If we let C represent the number of pounds of 20% cream that must be removed, and f the per cent of fat in the residue milk, we have

$$0.20\,C + f(Q - C) = qQ, \tag{1}$$

and

$$0.2713\,C + \frac{f(Q - C)}{0.27} = sQ. \tag{2}$$

From (1),

$$f = \frac{qQ - 0.20\,C}{Q - C}. \tag{3}$$

From (2),

$$f = \frac{0.27\,(sQ - 0.2713\,C)}{Q - C},$$

and hence if we equate these two values of f, we have

$$C = \frac{(q - 0.27\,s)\,Q}{0.12675},$$

or

$$C = 7.89\,Q\,(q - 0.27\,s). \tag{4}$$

After the value of C is found from formula (4), the value of f can be found from formula (3).

If the value of C obtained from formula (4) is negative, it simply means that the quantity of cream should be added, instead of removed, in order to obtain the required ratio of fat to solids. In this case the negative value of C should be used in solving for f in formula (3).

Illustrative Example. A manufacturer of powdered milk has 1000 lb. of natural milk which tests 4.25% and contains 12.78% total solids. How much 20% cream must be removed in order that the butter fat in the residue milk shall be 27% of the total solids? What is the per cent of fat in the residue milk?

In this case we have $Q = 1000$, $q = 0.0425$, $s = 0.1278$. Substituting these values in formula (4) on page 173, we have

$$C = 7.89 \times 1000 \times (0.0425 - 0.27 \times 0.1278)$$
$$= 63.072 \cdots.$$

Using the above value for C in formula (3), we then have

$$f = \frac{0.0425 \times 1000 - 0.20 \times 63.07}{1000 - 63.07}$$
$$= 0.03189 \cdots.$$

That is, 63.1 lb. of 20% cream must be removed to meet the required conditions, and the residue milk will test 3.19% fat.

Exercises. Conversion Formulas

In each of the following cases compute the amount of 20% cream to be removed (or added) in order that the fat content of the residue milk shall be 27% of the total solids, and determine the per cent of fat in the residue milk:

	Natural Milk	Butter Fat	Total Solids
1.	1000 lb.	3%	11.64%
2.	1400 lb.	$3\frac{1}{4}$%	11.87%
3.	1200 lb.	$3\frac{1}{2}$%	12.10%
4.	1600 lb.	$3\frac{3}{4}$%	12.33%
5.	500 lb.	4%	12.55%

	Natural Milk	Butter Fat	Total Solids
6.	800 lb.	$4\frac{1}{2}$%	13.01%
7.	1000 lb.	$4\frac{3}{4}$%	13.24%
8.	1500 lb.	5%	13.46%
9.	600 lb.	6%	14.38%
10.	400 lb.	7%	15.29%

11. A manufacturer mixes the following lots of milk: 800 lb. testing 3.2% fat and 11.82% total solids, 1200 lb. testing $4\frac{1}{2}$% fat and 13.01% total solids, and 1000 lb. testing 5.8% fat and 14.01% total solids. As in Exs. 1–10, how much 20% cream must be removed, and what is the fat content of the residue?

Exercises. Test on Chapter VIII

1. How many pounds of 3.7% milk are required in order to produce 40 lb. of 30% cream?

2. What price per pound is a dairyman receiving for butter fat when he sells 20% cream at $1.30 per gallon? 4% milk at 40¢ per gallon?

For the weights per gallon of milk and cream of various grades the table on page 349 should be consulted.

3. If 400 lb. of 5% milk yields 50 lb. of cream, what is the butter-fat content of the cream?

4. A farmer has 40,000 lb. of milk testing 4% fat. If he sells it as 25% sour cream and skim milk, receiving 60¢ per pound of butter fat for the cream and 30¢ per 100 lb. for the skim milk, what will be the gross receipts from the sale?

5. If, in Ex. 4, the farmer sells the milk as 20% sweet cream and skim milk, receiving $1.30 per gallon for the cream and 30¢ per 100 lb. for the skim milk, what will be the receipts?

6. Find the number of pounds of butter fat in the following shipment of milk: 460 lb. testing 4.20% fat, 520 lb. testing 4.00%, 900 lb. testing 3.25%, 150 lb. testing 6.00%, and 780 lb. testing 3.50%.

7. When the milk mentioned in Ex. 6 is separated, how many pounds of 30% cream will it make?

8. In Ex. 7, how many pounds of skim milk will there be? What will be the fat loss if the skim milk tests 0.06% fat? if it tests 0.01% fat?

9. How much 4% milk must be added to 80 lb. of 25% cream in order to reduce it to 20% cream?

10. How much 20% cream must be removed from 1200 lb. of 4.33% milk containing 12.68% total solids in order that the butter fat in the residue milk shall be 27% of the total solids? What will be the per cent of butter fat in the residue milk?

Exercises. General Review

1. Find the value of $ab^{\frac{1}{2}}$ and of $(ab)^{\frac{1}{2}}$ when $a = 25$ and $b = 144$; when $a = 36$ and $b = 121$.

2. The formula $c = \frac{1}{2}\, n(n-1)$ gives the total number of different connections c that can be made in a telephone exchange from which there are n lines running to subscribers' telephones. Find the number of connections when $n = 1500$; when $n = 5000$.

3. Solve the pair of equations $2x + y - 7 = 0$ and $3x - y = 3$, and check the results by drawing their graphs.

4. Find the value of 1.04^7 correct to five decimal places by means of the Binomial Formula.

5. Find the value of $\sqrt{\dfrac{33.0 \times 9.8 \times 6.8}{4.9 \times 2.7 \times 0.8}}$ by logarithms.

6. If there are 2500 bacilli in a glass jar, and if the bacilli increase by self-bisection once every 40 hr., what will be the number of bacilli at the end of 40 da.?

Use four-place logarithms and express the result as a number of four figures multiplied by some power of 10.

7. When money is worth 4%, what is the present value of a debt of \$3500 bearing interest at 5% and due in 5 yr.?

8. In how many years will an investment of \$400 amount to \$600 at 4%, compounded semiannually?

9. The formula $T = \dfrac{C + \frac{C}{n}}{t}$ expresses the business "turnover" T, where C is the amount of capital, $\frac{1}{n}$ is the fraction of a dollar that can be borrowed for each dollar of capital, and t is the number of months required for the turnover. Derive a formula for t in terms of C, n, and T.

10. Find the mean fat content in the following lots of milk: 450 lb. testing 4.20%, 500 lb. testing 4.25%, 900 lb. testing 3.25%, 200 lb. testing 6.25%, 800 lb. testing 3.65%.

CHAPTER IX

FARM MANAGEMENT

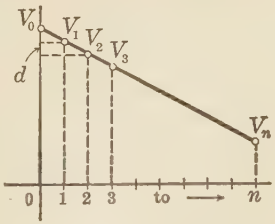
Depreciation. The decrease in value of a piece of property due to wear and age, or through becoming obsolete and failing longer to produce the required results, is called *depreciation.* For purposes of accounting, we assume a theoretical decrease per year, which is called the *annual depreciation.*

Straight-Line Method. A simple and common method of computing the annual depreciation is based upon the assumption that the depreciation occurs at a uniform rate throughout the life of the property, which we may take as n years. If we let V_0 be the original value and V_n the final value (the value at the end of n years), the total depreciation is evidently

$$V_0 - V_n.$$

A formula for d, the amount of the annual depreciation on the property, is therefore

$$d = \frac{V_0 - V_n}{n}.$$



Since the depreciated values V_1, V_2, V_3, $\cdots$, V_n decrease uniformly, the graph of these values is a straight line, as shown in the above figure. For this reason the method is called the *straight-line method* of calculating depreciation.

Briefly stated, the straight-line method consists in setting aside or "writing off" at the end of each of the n years in the life of the property one nth of the total depreciation. Although the actual depreciation may vary somewhat from this assumed figure, the simplicity of the method and the ease with which it may be applied in determining the value of property in making up an inventory recommend its use.

Average Depreciated Investment. When a man wishes to determine for a term of years the past actual cost or the future probable cost of his plant, such as stock, machinery, and buildings, it is much easier to use the average depreciated investment in computing the total interest and depreciation than to figure them for each year separately.

If we let V_0 be the original value (or investment), V_n the closing inventory at the end of n years, and d the annual depreciation, and employ the straight-line method, it is evident that the depreciated investment at the beginning of the second year is $V_0 - d$; at the beginning of the third year it is $V_0 - 2\,d$, and so on to $V_0 - (n-1)d$ at the beginning of the nth year.

For the average depreciated investment D we then have

$$D = \frac{V_0 + (V_0 - d) + (V_0 - 2\,d) + \cdots + V_0 - (n-1)d}{n}.$$

Since the numerator is an arithmetic series, we have

$$D = \frac{\frac{1}{2}\,n[V_0 + V_0 - (n-1)d]}{n} = \frac{V_0 + V_0 - nd + d}{2}.$$

But $$V_0 - nd = V_n;$$

whence $$D = \tfrac{1}{2}(V_0 + V_n + d).$$

In using this formula to calculate D during a term of k years, where $k < n$, it is evident that the final value V_n must be replaced by V_k, the value at the end of the kth year.

Since $V_1 = V_0 - d$, $V_2 = V_0 - 2\,d$, $V_3 = V_0 - 3\,d, \cdots$,

we have $$V_k = V_0 - kd.$$

For example, if a machine with a life of 10 yr. costs \$200, and its scrap value (that is, its final value) is \$20, what is the average depreciated investment during the first 6 yr.?

We first have $$d = \frac{V_0 - V_n}{n} = \frac{200 - 20}{10} = 18.$$

Then $$V_k = V_0 - kd = 200 - 6 \cdot 18 = 92;$$

and $$D = \tfrac{1}{2}(V_0 + V_k + d) = \tfrac{1}{2}(200 + 92 + 18) = 155.$$

That is, for the first 6 yr. the average depreciated investment in the machine is \$155.

Exercises. Depreciation

1. Taking the life of a house as 40 yr. and its scrap value as $200, find the annual depreciation on a house costing $5000.

2. In Ex. 1, what is the average depreciated investment in the house during its lifetime? the first 10 yr.? the last 10 yr.?

In the last question, V_0 should be taken as the value of the house at the beginning of the 30th year.

3. Taking the life of an office building costing $200,000 as 100 yr. and the scrap value as $10,000, find the average depreciated investment in the building during the 10-year period after the building is 40 yr. old.

4. If a horse 3 yr. old is worth $150, but is worth only $20 at the age of 20 yr., what is the annual depreciation? What is the rate of depreciation?

The rate should be expressed as a per cent of the original value.

5. In Ex. 4, what is the average depreciated investment between the ages of 3 yr. and 20 yr.?

6. If an automobile costing $2000 when new can be sold for only $350 when it is 5 yr. old, what is the average depreciated value of the machine during the 5 yr.? What is the annual rate of depreciation?

In actual practice the depreciation on automobiles is much greater the first year than in any subsequent year; in fact, it is often as high as 40 %. This is one objection to the use of the straight-line method. The special factors which influence the automobile business do not, however, affect the general theory of depreciation.

7. A farmer has a new house worth $5000, which has an estimated life of 40 yr.; a barn 5 yr. old, which was worth $2500 when new and has an estimated life of 20 yr.; a machine shed and a poultry house, each 3 yr. old, which were worth $500 each when new and have an estimated life of 15 yr. Taking the scrap value of the house as $200, that of the barn as $100, and those of the shed and poultry house as zero, find the average depreciated investment in the buildings during the next 5 yr.

Depreciation on Farm Machinery. The following table shows the average annual depreciation rates on several different classes of farm implements and machinery:

Binders, corn	8%	Harrows, disk	8%	Racks, hay	$10\frac{1}{2}$%
Binders, grain	8%	Loaders, hay	10%	Rakes, hay	7%
Cultivators, corn	7%	Mowers	$7\frac{1}{2}$%	Sleds	7%
Drills, grain	7%	Planters, corn	8%	Spreaders, manure	$10\frac{1}{4}$%
Engines, gas	$10\frac{1}{2}$%	Plows, gang	$7\frac{1}{4}$%	Tedders	$6\frac{1}{2}$%
Harness, heavy	$6\frac{1}{2}$%	Plows, walking	6%	Wagons	$5\frac{1}{2}$%

The above statistics were secured by the Minnesota Agricultural Experiment Station from several counties in that state. The rates are based upon the straight-line method of computing depreciation.

Exercises. Depreciation on Machinery

Using the above table, compute to the nearest 50¢ the value on April 1, 1935, of each of the following machines, which were purchased on April 1 of the specified year at the price stated:

1. Grain binder, 1925, $255.
2. Grain drill, 1927, $180.
3. Corn binder, 1929, $219.
4. Corn planter, 1929, $75.
5. Disk harrow, 1925, $100.
6. Mower, 1927, $85.
7. Hay loader, 1929, $122.
8. Hay rake, 1929, $44.
9. Gang plow, 1928, $119.
10. Farm wagon, 1928, $134.
11. Disk harrow, 1928, $121.
12. Hay rack, 1927, $15.
13. Corn cultivator, 1929, $57.
14. Walking plow, 1927, $25.
15. Manure spreader, 1928, $163.
16. Heavy harness, 1928, $75.
17. Gas engine, 1927, $91.
18. Hay tedder, 1926, $65.
19. Bob sled, 1925, $45.
20. Corn cultivator, 1925, $48.

21. Summarize the work of Exs. 1–20 on a ruled sheet, allowing one line to each item. In the first column place the name of the machine, in the second the year of purchase, in the third the purchase price, in the fourth the value on April 1, 1935, and in the last the average depreciated investment in the machine during the period.

Depreciation on Milch Cows. A study of the problem has shown that a formula for the approximate annual depreciation of milch cows is

$$d = 0.015\ V_0 + 0.16\tfrac{2}{3}(V_0 - V_n),$$

where V_0 is the original cost in dollars and V_n is the final "block" (or beef) value in dollars. In practice the block value is generally taken as \$30.

This formula was developed by the Minnesota Agricultural Experiment Station. The factor 0.015, or $1\frac{1}{2}$ %, is allowed for accidental death losses, and the factor $16\frac{2}{3}$ % is based upon a 6-year milking life.

For example, what is the approximate annual depreciation on a milch cow which originally cost \$150?

We have
$$\begin{aligned} d &= 0.015\ V_0 + 0.16\tfrac{2}{3}(V_0 - V_n) \\ &= 0.015 \times 150 + \tfrac{1}{6}(150 - 30) \\ &= 2.25 + 20 \\ &= 22.25. \end{aligned}$$

That is, the annual depreciation on the cow is approximately \$22.25.

Exercises. Depreciation on Milch Cows

1. Compute the annual depreciation for each cow in a group of 5 cows for which a farmer originally paid \$75, \$125, \$175, \$250, and \$500 respectively.

2. In Ex. 1 find the arithmetic mean annual depreciation for the entire group.

3. A dairyman has 3 cows valued at \$110 each, 5 cows valued at \$125, 4 cows valued at \$140, 7 cows valued at \$150, and 6 cows valued at \$175. Find the total annual depreciation for each of the five groups.

4. In Ex. 3 find the total annual depreciation for the entire herd and the mean annual depreciation per cow.

5. In Exs. 3 and 4 find the average depreciated investment for each group and for the entire herd during a period of 6 yr.

Weight and Volume Per Cents in Rations. Stock barns are much more likely to be provided with portable measures of volume, convenient for use at any point, than with portable scales. The per cents for the different ingredients in a ration are usually given with respect to weight, and it is therefore necessary to be able to reduce them to per cents with respect to volume.

In developing formulas for these reductions we may let W be the total weight of the ration; a_w, b_w, c_w, d_w, $\cdots$ the per cents by weight of the several ingredients; and k_a, k_b, k_c, k_d, $\cdots$ the weights of unit volumes of the corresponding ingredients. Similarly, we may let V be the total volume of the ration and a_v, b_v, c_v, d_v, $\cdots$ the required per cents by volume of the several ingredients.

The weights of the several ingredients are respectively

$$a_wW,\ b_wW,\ c_wW,\ d_wW, \cdots,$$

which are true for all values of W. For convenience, however, we may let $W = 100$, and hence the respective weights are a_w, b_w, c_w, and d_w.

The number of units of volume in the respective ingredients will therefore be

$$\frac{a_w}{k_a},\ \frac{b_w}{k_b},\ \frac{c_w}{k_c},\ \frac{d_w}{k_d}, \cdots,$$

and hence

$$V = \frac{a_w}{k_a} + \frac{b_w}{k_b} + \frac{c_w}{k_c} + \frac{d_w}{k_d} + \cdots. \qquad (1)$$

The number of units of volume in the respective ingredients will also be respectively

$$a_vV,\ b_vV,\ c_vV,\ d_vV, \cdots.$$

We therefore have the following formulas for the volume per cents a_v, b_v, c_v, d_v, $\cdots$:

$$a_v = \frac{a_w}{k_aV} \qquad (2) \qquad c_v = \frac{c_w}{k_cV} \qquad (4)$$

$$b_v = \frac{b_w}{k_bV} \qquad (3) \qquad d_v = \frac{d_w}{k_dV} \qquad (5)$$

Using the Formulas. The specifications for a certain ration call for 50 % corn, 25 % barley, 10 % shorts, and 15 % tankage by weight. The weights of unit volumes of these ingredients are as follows: corn, 6.5 lb.; barley, 5.6 lb.; shorts, 3.7 lb.; tankage, 4.1 lb. What are the per cents by volume?

Using formula (1) on page 182, we have $a_w = 50$, $b_w = 25$, $c_w = 10$, and $d_w = 15$, while $k_a = 6.5$, $k_b = 5.6$, $k_c = 3.7$, and $k_d = 4.1$. We therefore have

$$V = \frac{a_w}{k_a} + \frac{b_w}{k_b} + \frac{c_w}{k_c} + \frac{d_w}{k_d} = \frac{50}{6.5} + \frac{25}{5.6} + \frac{10}{3.7} + \frac{15}{4.1} = 18.51.$$

By applying formulas (2)–(5) on page 182, we then have

$$a_v = \frac{50}{6.5 \times 18.51} = 0.4156, \qquad c_v = \frac{10}{3.7 \times 18.51} = 0.1460,$$

$$b_v = \frac{25}{5.6 \times 18.51} = 0.2412, \qquad d_v = \frac{15}{4.1 \times 18.51} = 0.1977.$$

The percentages by volume are therefore as follows: corn, 41.56 %; barley, 24.12 %; shorts, 14.60 %; tankage, 19.77 %.

Check. 41.56 % + 24.12 % + 14.60 % + 19.77 % = 100.05 %, which is sufficiently close to check the accuracy of the solution.

These percentages can be applied to any unit of volume. For example, if the bushel is used, V will be 18.51 bu., and we shall require 41.56 % of 18.51 bu., or 7.69 bu. of corn, 4.46 bu. of barley, 2.70 bu. of shorts, and 3.66 bu. of tankage.

Exercises. Ration Per Cents

For each of the following rations find the per cents by volume for the different ingredients, when the per cents by weight and the weights of unit volumes of the ingredients are respectively as follows:

1. Corn, 90 %, 6.5 lb.; tankage, 10 %, 4.1 lb.

2. Corn, 50 %, 6.5 lb.; oats, 35 %, 3.7 lb.; alfalfa, 15 %, 0.6 lb.

3. Corn, 60 %, 6.5 lb.; oats, 8 %, 3.7 lb.; tankage, 32 %, 4.1 lb.

4. Corn, 60 %, 6.5 lb.; tankage, 10 %, 4.1 lb.; shorts, 30 %, 3.7 lb.

5. Corn, $66\frac{2}{3}$ %, 6.5 lb.; skim milk or buttermilk, $33\frac{1}{3}$ %, 9.3 lb.

Exercises. Miscellaneous Problems

1. The following figures were taken from a farmer's horse account for a year. *Investment:* 8 horses having a total value of $1200; harness and equipment, $280; barn, $1200. *Costs:* feed purchased, $576.86; labor, $140.50; upkeep of harness and equipment, $13.25; shoeing, $11.50; veterinary expenses, $10.30; insurance and taxes, $25.00; depreciation, 3% on barn and $6\frac{1}{2}$% on harness and equipment; interest, 6% on total inventoried investment at beginning of the year. *Horse Labor:* January, 324 hr.; February, 308 hr.; March, 512 hr.; April, 803 hr.; May, 1148 hr.; June, 690 hr.; July, 930 hr.; August, 1080 hr.; September, 1214 hr.; October, 835 hr.; November, 640 hr.; December, 280 hr. Find for the year the average cost of horse labor per hour.

2. In Ex. 1 find the per cent which each of the following costs is of the total cost: feed, labor, interest on investment, shelter (depreciation on barn), and all other costs. Draw a bar graph to show how the total cost is distributed.

3. A full working month for a horse is generally taken as 26 da. of 8 hr. each. In Ex. 1 find for each month and also for the entire year the average number of hours per day worked by each horse.

4. In Ex. 3 compute for each month the per cent of efficiency (the ratio of the number of hours worked to the number of hours in the working month) of horse labor. Find also the average per cent of efficiency for the year.

5. Draw a bar graph showing the monthly and yearly efficiency of horse labor, as determined in Ex. 4.

6. At the last inventory the value of a 14-inch walking plow, which originally cost $25, was $19, and the repairs for the year amounted to $1.70. Allowing annual depreciation at 6% and interest on the last inventory value at 6%, find the cost per acre chargeable against the use of the plow on 28 A. for the year.

7. The following table shows the original cost of several farm machines, their value at the last inventory, the rate of annual depreciation, the expenses during the year for repairs and labor upon the machines, and the number of acres worked on a farm in which the total number of crop acres is 120:

Machine	Original Cost	Present Inventory Value	Rate of Depreciation	Depreciation Charge	Labor and Repairs	Interest Charge	No. of Acres	Charges per Acre
Grain Binder	$225	$171	8%		$3.40		60	
Drill	180	126	7%		2.90		60	
Disk Harrow	120	90	8%		.60		90	
Corn Binder	220	154	8%		5.40		30	
Mower	85	51	7½%		4.15		30	

Copy the table and insert in the depreciation-charge column the amount of the annual depreciation, in the interest column the charges at 6% on the value at the last inventory, and in the last column the cost per acre (including interest, depreciation, labor, and repairs) chargeable to each machine for the year.

8. In Ex. 7 find the total machine-depreciation charge per crop acre for the year and the total machine charge (depreciation, interest, labor, and repairs) per crop acre for the year.

9. A farmer pays $216 for a grain binder which has an estimated life of 12 yr. and the final value zero. Compute the annual depreciation charges and the interest charges at 6% on the initial inventoried value each year for the first 5 yr.

10. In Ex. 9 repairs are estimated at $1 for the first year with an increase of $4 for each additional year. Find the amount chargeable to binder cost for each of the 5 yr.

11. If in Exs. 9 and 10 the farmer cuts 60 A. each year, what is the binder cost per crop acre for each of the 5 yr.?

12. In Exs. 9–11 find the average depreciated investment and the average binder cost per crop acre for the 5 yr.

13. The following statistics show a year's record of a herd of 8 cows. *Investment* (at last inventory): 8 cows, valued as shown in the table below; 1 bull, \$150; dairy utensils, \$25;

Cow . . .	1	2	3	4	5	6	7	8
Value . .	\$90	\$85	\$80	\$100	\$100	\$80	\$90	\$85
Feed . .	\$67.20	\$71.30	\$64.80	\$70.75	\$68.50	\$62.50	\$73.90	\$66.00
Milk (lb.)	7910	6420	5208	8220	7380	6400	7500	5420
Fat . . .	3.6%	3.9%	4.3%	3.8%	4.0%	4.5%	3.9%	4.1%

barn, \$1000. *Costs:* feed for cows, as in table; feed for bull, \$65; labor for cows, \$256; labor for bull, \$20; insurance, \$10; veterinary expenses and spraying, \$5.50; repairs and taxes, \$22; interest on cows, bull, and utensils, 6%; interest on barn, 5%; depreciation on cows, as on page 181; depreciation on barn, \$30. *Products:* milk, as in table; 8 calves at \$5 each; 90 T. of manure at \$1 per ton at the barn. The butter fat was sold at 55¢ per pound, and the skim milk (80% of the whole milk) at 30¢ per 100 lb. Crediting the value of all other products against the expenses, compute the average cost per pound of producing the butter fat.

14. In Ex. 13 find the labor cost for each cow.

First find the average cost per cow, and then reduce this result by 50¢ per 1000 lb. of milk below the average production and increase it by 50¢ per 1000 lb. above the average, computing only to the nearest 1000 lb.

15. In Ex. 13 find the value of the manure that should be credited to each cow.

First deduct $\frac{1}{9}$ for the bull, and then find the average production per cow. Reduce or increase this amount 50¢ per 1000 lb. of milk below or above the average production.

16. In Ex. 13 find by means of six-place logarithms the average gain or loss per pound of butter fat produced.

Good practice would require replacing cows 3 and 8, which are low producers, with cows producing more than 6000 lb.

Farm Survey. The following outline, which is typical of a survey of the business of the ordinary general farm, is to be used as the basis for the problems on page 188.

Land. Value, $24,000. Acreage, 320, apportioned as follows: waste areas, 32 A. of sloughs; necessary nonproducing areas, 20 A., including 8 A. for highways, 6 A. for farmstead, and 6 A. for fixed roads and fences; production areas, 268 A., including 28 A. of pasture and 240 A. for crops (wheat, corn, clover, and oats, 60 A. each per year).

Buildings and Machinery. Buildings (new): value, $10,000; life, 25 yr. Machinery (new): value, $3000. General building equipment: value, $1000.

Live Stock. 6 horses having a total value of $850; 100 fowls valued at $1 each; 10 sows valued at $30 each; 15 dairy cows valued at $100 each.

Annual Costs. Total man labor (including field and household help and wife's wages), $2100; general live-stock expense (exclusive of labor), $225, including $50 for horses, $25 for poultry, $50 for swine, $100 for cows; total feed purchased, $450; insurance, $50; taxes (not including income taxes), $300; interest on investment, 6%; depreciation on machinery, 8%; depreciation on general building equipment, 10%; depreciation on buildings, as computed by the straight-line method; depreciation on dairy cows, as computed from the formula on page 181.

Live-stock depreciation is allowed only on the dairy cows, which may be assumed to have a milking life of 6 yr. and a final block value of $30. The other live stock is assumed to hold its value steady by breeding.

Since all the corn, oats, and clover raised on the farm are consumed by the stock, the cash outlay for feed is kept down to the small amount listed.

Annual Products. 125,000 lb. of market milk at $2.60 per 100 lb.; 15 calves at $5 each; poultry products, $400; 140 pigs, each developing a market weight of 200 lb. worth 8¢ per pound; 20 bu. of wheat per acre at $1.25 per bushel.

Exercises. Farm Survey

Using the data on page 187, solve the following problems:

1. Considering the total investment each year as the total inventoried value at the close of the preceding year, find the amount of the interest-on-investment charge for each of the first 5 yr.; for the entire 5-year period.

2. The depreciation fund is deposited in a savings bank at the close of each year. If it draws interest at 4%, compounded semiannually, and remains untouched until the end of the fifth year, what is the amount of the fund at this time?

3. Find for each of the 5 yr. and for the entire period the per cent which building expense is of the total expense.

4. As in Ex. 3, find for each year and for the entire period the per cent of the total expense for (1) labor cost, (2) depreciation allowances, and (3) all other expenses.

5. Taking the total acreage of the farm, find the building charge per acre for each year and for the entire period.

6. Between Sept. 15 and Dec. 15, the date on which the young pigs were marketed, they gained an average of $1\frac{1}{4}$ lb. per day. Allowing 6 lb. of corn weighing 56 lb./bu. for each pound of gain in weight, find the number of bushels of corn required for the pigs between these dates.

7. Find the net income of the farm for each of the 5 yr. and for the entire period.

8. Taking the total acreage of the farm, find the income per acre for each of the 5 yr. and for the entire period.

9. Find the net income per crop acre for each of the 5 yr. and for the entire period.

10. Find the per cent of profit on the investment for each of the 5 yr.

11. Find the per cent of profit on the average depreciated investment for the entire 5-year period.

Exercises. Test on Chapter IX

1. If a machine costing \$1200 has a life of 8 yr. and a scrap value of \$80, what is the annual depreciation allowance?

2. If the life of a barn that cost \$8000 is 25 yr., after which it is worthless, what is the average depreciated investment for the first 12 yr. of its life?

3. Compute the annual depreciation for each of three cows valued respectively at \$90, \$130, and \$160.

Use the formula given on page 181.

4. Find the average annual depreciation of the three cows mentioned in Ex. 3.

5. The following table shows part of a schedule giving the accrued depreciation for a machine which cost \$2000 when new, and which has a life of 9 yr. and a scrap value of \$200:

Age in Years	Value at End of Year	Depreciation Allowance	Accrued Depreciation
1	\$1800	\$200	\$200
2	1600	200	400
3	1400		
4			
5			
6			
7			
8			
9			

Copy and complete the table. Draw a bar graph to represent the facts set forth by the table, using bars with white sections to show the depreciated values and shaded sections to show the accrued depreciation.

6. It takes 450 lb. of a ration containing 92%, by weight, of corn weighing 56 lb./bu. and 8% of tankage weighing 35.3 lb./bu. to produce 100 lb. of pork. Find the volume of each ingredient needed daily for a pig which gains $1\frac{1}{2}$ lb. per day.

Exercises. General Review

1. How long will it take \$6000 to amount to \$9000 at 4%, compounded annually?

2. In physics it is learned that the density D of a substance is the mass M in grams divided by the volume V in cubic centimeters. Express this rule by a formula.

3. In the formula of Ex. 2 find D when $M = 354$ and $V = 100$; when $M = 184.46$ and $V = 30$.

Find the prime factors of the following expressions:

4. $x^2y + xy^2 + xy$.
6. $x^3 - 27\,y^3$.
8. $2\,x^2 - 8\,y^2$.
5. $6\,x^2 + 20\,x - 16$.
7. $x^4 - 81$.
9. $4\,x^2 + 16\,x - 84$.

10. State one condition under which $l + m + n = lmn$.

11. State one condition under which $lmn = \frac{lm}{n}$.

12. Why do we often plot three points in determining the graph of a linear equation?

13. What is meant by saying that an equation in two variables has an indefinite number of solutions?

Without solving, determine whether or not the roots of the following equations are as stated:

14. $x^2 - 5\,x + 6 = 0$; 2, 3.
15. $x^2 - 8\,x + 15 = 0$; 5, -3.
16. $x^2 + x - 12 = 0$; -3, 4.
17. $y^2 + 12\,y + 35 = 0$; 5, 7.
18. $y^2 - 6\,y - 9 = 0$; 3, -3.
19. $x^2 + abx - ab = 0$; a, b.

20. On Jan. 1, 1905, a boy deposited \$20 in a savings bank which pays interest at 4%, compounded semiannually. If he should forget all about the deposit until Jan. 1, 1935, what amount would he then have to his credit?

21. The consumption C of coal, in tons per hour, in a locomotive is given by the formula $C = 0.005\,v^2$, where v represents the speed in miles per hour. Draw a graph of this formula.

CHAPTER X

VARIATION

Direct Variation. In the study of functions in Chapter III we learned how one variable may depend upon another for its value. If two variable quantities x and y have a constant ratio, either quantity is said to *vary as the other* or to *vary directly as the other*. If we let k represent a constant, this type of variation, which is called *direct variation*, may be represented by any one of the following expressions:

$$\frac{x}{y} = k, \quad x = ky, \quad y = \frac{x}{k}.$$

For example, in the formula $A = \pi r^2$, which gives the area A of a circle in terms of the radius r, the symbol π represents the constant having the approximate value $3\frac{1}{7}$. Since this formula is of the type $x = ky$, it is evident that we have a case of direct variation. In this case the area is seen to vary directly as the square of the radius.

Inverse Variation. If a variable x varies as the reciprocal of a variable y, x is said to *vary inversely as* y. This type of variation, which is called *inverse variation*, may be represented by any one of the following expressions:

$$\frac{x}{\frac{1}{y}} = k, \quad xy = k, \quad x = \frac{k}{y}, \quad y = \frac{k}{x}.$$

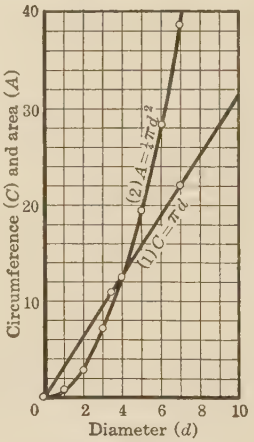
For example, when d is taken as a constant, the formula $d = rt$, which gives the distance d traveled in the time t by a body moving at the rate r, is of the form $xy = k$, and hence is a case of inverse variation. That is, over a fixed distance the time decreases as the rate increases. For example, at 20 mi./hr. it will take 5 hr. to go 100 mi., while at 40 mi./hr. it will take only $2\frac{1}{2}$ hr., or half as long. Conversely, to cover a distance of 100 mi. in 4 hr. will require a speed of 25 mi./hr., while to cover the same distance in 8 hr. will require a speed of only $12\frac{1}{2}$ mi./hr.

Joint Variation. A variable is said to *vary jointly as two or more others* if it varies directly as their product.

For example, the area A of a rectangle is bh, the product of the base b and the height h. That is, the formula $A = bh$ represents a case of joint variation in which A varies jointly as b and h.

Graphs Illustrating Variation. Since π is a constant, the two formulas $C = \pi d$ and $A = \frac{1}{4}\pi d^2$ involve cases of direct variation; that is, we have

$$\frac{C}{d} = \pi \quad \text{and} \quad \frac{A}{d^2} = \tfrac{1}{4}\pi.$$

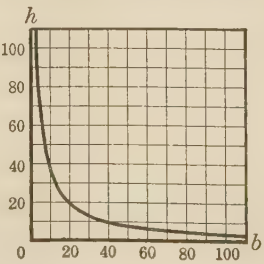


If we draw the graphs of these two formulas to the same scale on the same axes, as shown in the figure at the right, we see that an increase in d always results in an increase in either C or A. Obviously, the rates of increase are not the same for C and A, since C varies directly as the first power of d while A varies directly as the square of d.

If we wish to construct a rectangle with an area of 400 sq. in., we have a case of inverse variation involving the equation

$$bh = 400.$$

If $b = 40$ in., then $h = 10$ in.; if $b = 20$ in., then $h = 20$ in.; and so on, a decrease in b leading to an increase in h, and conversely. These relations are more clearly shown by the graph at the right.



By locating points corresponding to negative values of b and h in the equation $bh = 400$ we can plot the second branch of the hyperbola (page 62) which forms the complete graph of an equation of the type $xy = k$. This case of the hyperbola is often called a *rectangular hyperbola*. In the above case it is evident that negative values for b and h have no practical significance, and the same is true of negative values in most practical problems.

Applications of Variation. The following problems will serve to illustrate practical applications of the different types of variation which have been mentioned:

1. In physics it is shown that the pressure in a body of fluid varies directly as the depth. If the pressure exerted on the bottom of a vessel by a column of water 2 ft. high is 0.868 lb./sq. in., how deep must the water be to exert a pressure of 22 lb./sq. in.?

Letting P represent the pressure in pounds per square inch, and d the depth in feet, we have

$$\frac{P}{d} = k;$$

whence

$$k = \frac{0.868}{2} = 0.434.$$

Hence, when the pressure is 22 lb./sq. in., we have

$$d = \frac{P}{k} = \frac{22}{0.434} = 50.691 \cdots;$$

that is, it will require a column of water 50.69 ft. high.

2. The safe center load for a horizontal beam supported at both ends varies jointly as the breadth and the square of the depth of the beam and inversely as the length between the supports. If the maximum safe load for a 2×8 yellow-pine timber 12 ft. long is 1067 lb., what is the maximum safe load for a 3×12 yellow-pine timber 14 ft. long?

In connection with lumber the expression 2×8 means 2 in. in breadth and 8 in. deep, and similarly for 3×12.

Letting S represent the safe load in pounds, b the breadth in inches, d the depth in inches, and L the length in feet, we have

$$\frac{S}{bd^2 \cdot \frac{1}{L}} = k;$$

whence

$$k = \frac{SL}{bd^2} = \frac{1067 \cdot 12}{2 \cdot 8^2} = 100.031 \cdots.$$

Then

$$S = \frac{kbd^2}{L} = \frac{100.03 \times 3 \times 12^2}{14} = 3086.64.$$

That is, the maximum safe load is approximately 3087 lb.

Exercises. Variation

1. If x varies directly as y, and if $x = 2$ when $y = 5$, what is the value of y when $x = 3.6$?

As in the cases on page 193, we must first find the value of k in the expression $\frac{x}{y} = k$. In this case we have $k = \frac{2}{5}$, and hence we can easily find the value of y when $x = 3.6$.

2. If x varies inversely as z, and if $x = 3$ when $z = 8$, what is the value of z when $x = 5$?

3. In Ex. 2, what is the value of x when $z = 16$?

4. If V varies directly as T and inversely as P, and if $V = 100$ when $T = 300$ and $P = 15$, what is the value of V when $T = 360$ and $P = 30$? What value must P have in order that V shall remain constant?

5. If PV is constant, and if $P = 50$ when $V = 4$, what is the value of V when $P = 320$?

6. The circumferences of circles vary directly as their diameters. What is the speed of a pulley $5\frac{1}{2}$ in. in diameter that is driven by a pulley which is 24 in. in diameter and makes 60 revolutions per minute?

7. If a wagon wheel 4 ft. 4 in. in diameter makes 372 revolutions in traveling a fixed distance, how many revolutions will a wheel 5 ft. 4 in. in diameter make in traveling the same distance?

8. It is proved in geometry that the areas of similar plane figures vary directly as the squares of corresponding dimensions. If two circular metallic plates, each 1 in. thick and with diameters of 6 in. and 8 in. respectively, are melted and cast into a single circular plate 1 in. thick, what is the diameter of this plate?

9. What is the radius of a single circle whose area is equal to the sum of the areas of three circles having radii of 3 in., 4 in., and 5 in. respectively?

10. If a metal disk with a radius of 3 cm. has a mass of 50 g. (grams), what will be the radius of a smaller disk cut concentrically from the larger one in such a way that the ring of metal remaining will have the same mass as the disk cut out?

11. It is proved in geometry that the volumes of similar solids vary directly as the cubes of corresponding dimensions. If the weight of a metal ball having a radius of 2 in. is 6 lb., what is the weight of a ball made of the same kind of metal and having a radius of 4 in.?

12. The density (mass divided by volume) of the moon is approximately $\frac{8}{10}$ that of the earth, and the radius of the moon is approximately $\frac{1}{4}$ that of the earth. Find the ratio of the mass of the moon to that of the earth.

13. It is desired to cut a stick of timber that shall contain eight times as many cubic inches of wood as a 2×4 joist 12 ft. long. If all dimensions are to be increased proportionally, what dimensions should be allowed for the stick?

As on page 193, the expression 2×4 means 2 in. wide by 4 in. deep.

14. What would be the size of the timber mentioned in Ex.13 if the length were to be kept the same; that is, 12 ft.?

15. The theoretical velocity (in feet per second) of the flow of water through an opening varies directly as the square root of the height (in feet) of the water column causing the flow. Taking the constant of variation as 8.02, find the height of the column needed to cause a flow of 30 ft./sec.

16. If the velocity of the flow of water through an opening is 48 ft./sec. when the head is 36 ft., what is the velocity of flow when the head is 64 ft.?

17. It has been found that the carrying power of a swift stream varies as the sixth power of the speed of the water. If the speed is checked one half, what is the effect on the carrying power? What is the effect when the speed is increased one half? when the speed is doubled?

18. Observations of the average temperature and the rainfall in North Dakota covering a series of years seem to indicate that the yield per acre of spring wheat varies approximately as the May and June rainfall and inversely as the fourth power of the mean June temperature. If the yield is 15 bu. per acre when the May and June rainfall is 7 in. and the mean June temperature is 58°, what is the probable yield when the May and June rainfall is 4 in. and the mean June temperature is 66°?

19. The amount of simple interest received from an investment varies jointly as the principal, rate, and time. Write this law as a formula, letting p represent the principal, i the interest, r the rate, and t the time in years.

20. The time of vibration of pendulums in the same locality varies directly as the square root of the length. If the time of vibration of a pendulum 40 in. long is 7 sec., what will be the time of vibration of a pendulum 20 in. long? What length must a pendulum have in that locality in order to vibrate once in 1 sec.? in 2 sec.? in 3 sec.?

21. The theoretical efficiency of a steam engine varies directly as the change in temperature of the live steam between boiler and condenser and inversely as the temperature of the live steam at the boiler. If the efficiency E is 25% when the temperature T_1 of the live steam is 400° and the temperature T_2 at the condenser is 300°, what must the temperature at the condenser be in order to give an efficiency of 33%? If the temperature at the condenser is 330°, what must be the temperature of the live steam in order to maintain the efficiency of the engine at 25%?

22. The amount of illumination given by a light varies directly as the intensity of the light and inversely as the square of the distance from the light. How far from a screen must a light be placed in order to give four times as much illumination as is given by another light of only half the intensity and 25 ft. from the screen?

Exercises. Test on Chapter X

Write the numbers 1 to 15 in a column on your paper and after each number write "direct" or "inverse" according as the corresponding formula below indicates direct or inverse variation:

1. $A = \pi r^2$.
2. $PV = k$.
3. $C = \pi d$.
4. $x = ky$.
5. $y = \frac{x}{k}$.
6. $d = rt$.
7. $V = \frac{4}{3}\pi r^3$.
8. $s = \frac{1}{2}gt^2$.
9. $T = 6e^2$.
10. $x = \frac{k}{y}$.
11. $A = \frac{1}{4}\pi d^2$.
12. $d = 1.22\sqrt{h}$.
13. $C = 4w + 5$.
14. $F = 1.8C + 32$.
15. $t = \pi\sqrt{\frac{l}{980}}$.

In Ex. 6, d may be regarded as a constant, and in Ex. 8, g represents the constant acceleration of gravity.

16. If a varies jointly as b^2 and c, and if $b = 5$ when $a = 6$ and $c = 9$, what is the value of a when $b = 2$ and $c = 3$? Write the equation expressing the relation of a, b, and c.

17. Find three numbers which vary directly as 2, 3, and 6 and the sum of whose squares is 1.

18. Wind pressure on a plane surface varies jointly as the area of the surface and the square of the velocity of the wind. If a wind velocity of 32 mi./hr. results in a pressure of 4 lb./sq. ft., what wind velocity is necessary to give a total pressure of 252 lb. on a sail having an area of 28 sq. ft.?

19. The volume of a body of gas varies directly as the absolute temperature and inversely as the pressure exerted upon the gas, a relationship which may be expressed by the formula $V = \frac{RT}{P}$, or $PV = RT$, where V is the volume, P the pressure, T the absolute temperature, and R the constant of variation. If the volume of a body of gas is 50 cu. cm. (cubic centimeters) when the temperature is 0° C. (273° on the absolute scale) and the pressure is 8 kg. (kilograms), what is the pressure when the volume is 65 cu. cm. and the temperature is 15° C.?

Exercises. General Review

1. From the formula $\frac{E}{e} = \frac{R+r}{r}$ find the value of r to the nearest 0.1 when $E = 7.6$, $e = 2.3$, and $R = 52.7$.

2. Two partners, A and B, divide the profits of their business so that A, who owns the building in which they do business, receives 30% more than B. How much does each partner receive in a year in which the profits amount to \$20,700?

3. If $a = \sqrt{2}$, $b = \sqrt{5}$, and $c = \sqrt{7}$, what to the nearest 0.01 is the value of $a^2 + b$? of $a^3 - 2$? of $b^2 - c$? of $a^2 + b^2 + c$?

4. Show how you would find the logarithms of 6, $\frac{2}{3}$, and $\sqrt[3]{12}$ if you know the logarithms of 2 and 3.

5. Without expanding, find the sum of the coefficients in the expansion of the binomial $(3x - 2y)^9$.

6. Find the eleventh term and the sum of the first eleven terms of the series $1, \frac{3}{2}, 2, \cdots$.

7. Show how to divide \$660 among three persons in such a way that the first has four times as much as the second, and the second has twice as much as the third.

8. Find the amount of \$1800 placed at interest for 12 yr. at 5%, compounded semiannually.

9. A machine costing \$12,000 is estimated to have a life of 10 yr. and to be worth \$1000 at the end of this time. By the straight-line method, what is the annual depreciation charge? At 5%, compounded annually, what annual payment must be made into a sinking fund that will replace the machine at the end of the 10 yr.?

10. How many pounds of $3\frac{1}{2}\%$ milk and 20% cream will 1800 lb. of 5.6% milk yield?

11. If a 3×10 beam 15 ft. long, supported at both ends, will carry a center load of 2000 lb., what center load can be carried by a 4×12 beam 18 ft. long and similarly supported?

CHAPTER XI

NUMERICAL TRIGONOMETRY

Indirect Measurement. Although it is possible to measure many objects directly by using a foot rule, a yardstick, a meter stick, a steel tape, or some similar instrument, many measurements, such as the distance across a wide river, the distance across a swamp, the height of a mountain, and so on, must be made indirectly. Indirect measurements of this sort are frequently made by measuring the parts of triangles and computing the required distances by *trigonometry*. For this reason the triangle occupies an important position in mathematics.

The word "trigonometry" comes from three Greek words meaning "three," "angle," and "measure," or "triangle measure."

Another reason for the importance of the triangle is that it is used extensively in all types of construction, such as houses, bridges, ships, and so on.

In this figure of a bridge, for example, the steel work is arranged in the form of triangles. The triangle is a rigid figure; that is, a triangle will keep its shape, even though the three sides may be merely pivoted together.

A third reason for the importance of the triangle is that we can find the area of any plane rectilinear figure by dividing it into triangles and finding the area of each one.

Even if the whole figure is not divided into triangles, there are generally one or more parts which are triangular in shape.

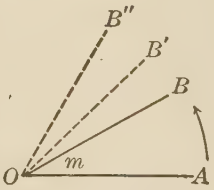
In this chapter we shall see how the triangle may be used as the basis of different kinds of measurements by applying trigonometric methods.

Lines and Angles. Before going further we shall briefly review the important geometric terms that we shall need to use.

Since *straight*, *curve*, and *parallel lines* are so familiar, further definitions are not necessary. If we wish to speak of a definite part of a line, or of a line limited by two points, we call it a *line segment*, or simply a *segment*.

In general, we use the word "line" to mean a straight line, and we assume that a straight line can be extended indefinitely in either direction.

We ordinarily think of an *angle* as formed by any two lines that proceed from the same point, as in the case of the lines OA and OB proceeding from the point O here shown. The point O is called the *vertex*, and OA and OB are called the *arms*, or *sides*.

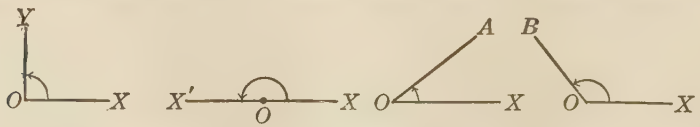


We may refer to this angle as "angle AOB" (usually written $\angle AOB$, with the vertex letter in the middle), as "angle O," or simply as m, according to which is the most convenient.

In the above figure we may regard $\angle AOB$ as generated by revolving the line OB about O from an initial position OA to the terminal position OB. If OB continues to revolve, as shown by the dotted lines, the angle increases; that is, the size of an angle depends upon the amount of turning.

For example, $\angle AOB'$ is greater than $\angle AOB$ (written $\angle AOB' > \angle AOB$), and $\angle AOB$ is less than $\angle AOB''$ (written $\angle AOB < \angle AOB''$).

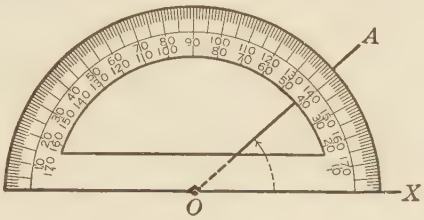
The four figures shown below illustrate respectively a *right angle*, a *straight angle*, an *acute angle*, and an *obtuse angle*.



If one line forms a right angle with another line, each line is *perpendicular* to the other, and if the lines form an acute or an obtuse angle, each line is *oblique* to the other.

For example, in the figure at the left, above, OX is perpendicular to OY.

Measurement of Angles. In measuring the sizes of angles in plans or maps we often use a *protractor* like the one here shown. A complete circle is divided into 360 units, called *degrees* (°), and hence a straight angle contains 180°, and a right angle contains 90°.

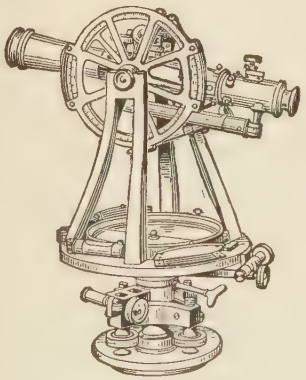


In this figure, $\angle XOA$ contains approximately 42°.

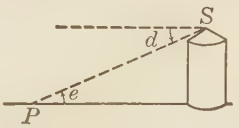
If the sum of two angles is 90°, each angle is said to be the *complement* of the other; if the sum is 180°, each angle is said to be the *supplement* of the other.

The degree is further subdivided into 60 minutes (′), and the minute into 60 seconds (″), or else decimal subdivisions of the degree are used.

Surveyors and others who have occasion to measure angles out of doors generally use a *transit* like the one here shown. The scale on the left of the telescope measures the angle through which the telescope is moved up or down in sighting at an object. The scale on the circle at the base of the supports of the telescope measures the angle through which the telescope is moved to the right or left. With a good transit it is possible to measure angles to a high degree of accuracy.



The angle between the line of sight to an object and the horizontal is called an *angle of elevation* or an *angle of depression* according as the object is situated above or below the level of the eye.

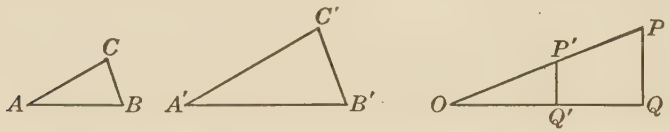


Thus, in this figure, e is the angle of elevation of S observed from P, and d is the angle of depression of P observed from S.

Geometry of the Triangle. In our work with triangles in trigonometry we shall need to use several facts which are proved in geometry, but which we do not need to prove here.

Similar Triangles. The first fact relates to *similar triangles*, which are triangles that have the same shape but not necessarily the same size, and may be stated as follows:

Similar triangles have their corresponding angles equal and their corresponding sides proportional.



For example, if the triangles ABC and $A'B'C'$ (usually written $\triangle ABC$ and $A'B'C'$) shown above are similar, we have

$$\angle A = \angle A',\ \angle B = \angle B',\ \text{and}\ \angle C = \angle C'.$$

Also,
$$\frac{AB}{A'B'} = \frac{BC}{B'C'},\ \frac{BC}{B'C'} = \frac{CA}{C'A'},\ \text{and}\ \frac{CA}{C'A'} = \frac{AB}{A'B'};$$

whence
$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

Similar Right Triangles. In the case of two right triangles we shall also need to use the fact that

Two right triangles which have an acute angle of one equal to an acute angle of the other are similar.

For example, in the right triangles OQP and $OQ'P'$ shown above, $\angle O$ is the same in each triangle, and hence the two triangles are similar.

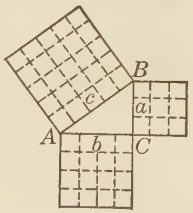
The student is, of course, familiar with the customary classification of triangles according to their angles as *equiangular*, *acute*, *obtuse*, and *right*, and according to their sides as *equilateral* or *isosceles*.

Angle Sum of a Triangle. A third important fact, which relates to the sum of the angles of a triangle, is that

The sum of the angles of any triangle is 180°.

It follows from this that the sum of the acute angles of a right triangle is always 90°; that is, the two acute angles are complementary.

Pythagorean Theorem. Another geometric fact of great importance deals with the relation between the hypotenuse and the sides of a right triangle. This relation, which is believed to have been first proved by Pythagoras, a Greek mathematician of about 550 B.C., may be stated as follows:



The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

This relation is easily remembered by means of the formula

$$c^2 = a^2 + b^2,$$

in which c represents the hypotenuse and a and b the sides.

Exercises. Triangles

1. If the sides of a triangle are 3 in., 3 in., and 4 in. respectively, what is the perimeter of a similar triangle in which the longest side is 6 in.?

2. In drawing a map to the scale of $\frac{1}{100000}$, what lengths, to the nearest 0.1 in., should be taken for the sides of a rectangular county 30 mi. long and 20 mi. wide?

It is obvious that a map, which is a scale drawing, is similar in outline to the territory which it represents.

3. How many degrees are there in each angle of an equilateral triangle?

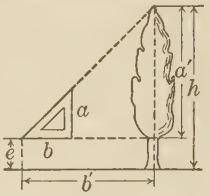
4. If the angle between the equal sides of an isosceles triangle is 40°, how large is each of the other angles?

5. Write a formula for the height h of an equilateral triangle in terms of the side s.

6. Write a formula for the side s of an equilateral triangle in terms of the height h.

7. Find the length of the hypotenuse of a right triangle having sides of 3 in. and 4 in.; of 5.6 in. and 8.2 in.

8. This figure, which is not drawn to scale, shows how we might find the height of a tree by holding a right triangle, in which $a = b$, at the distance e above the ground. If $e = 5$ ft. 4 in., and $b' = 23$ ft. 8 in., how high is the tree?



Since $a = b$, how do a' and b' compare? Why?

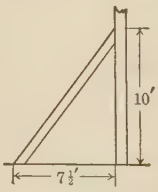
9. For all practical purposes the sun's rays may be regarded as parallel when they reach the earth. If a flagpole casts a shadow 78 ft. long at the same time that a post 5 ft. high casts a shadow 6 ft. long, how high is the flagpole?

In all such cases the objects are assumed to stand vertically upon a horizontal plane unless it is expressly stated otherwise. Draw a simple figure, not necessarily to scale, to illustrate the problem.

10. If a telephone pole casts a shadow 30 ft. long at the same time that a post 5 ft. high casts a shadow 4 ft. long, how tall is the pole?

11. How long is the shadow cast by a yardstick at the same time that a foot rule casts a shadow 9 in. long?

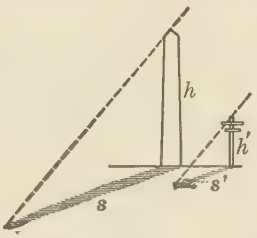
12. In bracing a vertical timber a carpenter wishes the brace to run from a point 10 ft. above the floor to a point $7\frac{1}{2}$ ft. from the foot of the timber. In finding how long a brace is needed he lays his square flat on the floor and marks one point on the long arm 10 in. from the corner and a second point on the short arm $7\frac{1}{2}$ in. from the corner. He then takes the number of inches between these two points as the length of the brace in feet. Does this give the correct length? State the reasons for your answer.



13. Using the method of Ex. 12, find how long a slanting brace is needed to reach from a point 6 ft. from the foot of a timber to a point 8 ft. above the floor.

If a carpenter's square is not available, two rulers placed at right angles to each other will do as well.

Trigonometric Ratios. We have seen how to find the height of an object by using the ratios between the corresponding sides of similar triangles. For example, we can find the height of the monument here shown if we know the height h' of the telegraph pole, the length of the shadow s of the monument, and the length of the shadow s' of the pole. That is, since the triangles are similar, we have



$$\frac{h}{h'} = \frac{s}{s'},$$

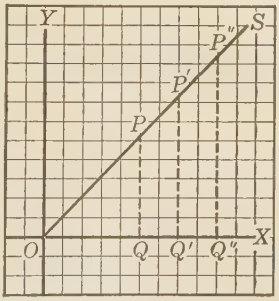
or

$$h = \frac{h'}{s'} \cdot s,$$

and hence we can find the value of h.

It should be observed, however, that the height of the pole is relatively unimportant. If the pole were twice as high, its shadow would be twice as long; if it were half as high, its shadow would be only half as long, and so on. That is, the ratio of height to shadow is always the same for any given angle at which the sun's rays strike the earth.

For example, in this figure we may let SO represent the sun's rays, QP, $Q'P'$, and $Q''P''$ poles of various heights, and OQ, OQ', and OQ'' the corresponding shadows. We then have



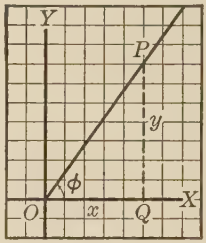
$$\frac{QP}{OQ} = \frac{Q'P'}{OQ'} = \frac{Q''P''}{OQ''};$$

that is, for any given angle, as $\angle XOS$, the ratio of the ordinate of a point on the terminal side to the abscissa of the point is always the same.

Besides the ratio of ordinate to abscissa (the ratio $\frac{QP}{OQ}$ in the second figure), we may take other ratios, such as $\frac{QP}{OP}$, $\frac{OQ}{OP}$, and so on. These ratios are called *trigonometric ratios*, and we shall now see how they are used in indirect measurements.

Tangent of an Angle. The different trigonometric ratios are given special names. For example, the ratio of ordinate to abscissa, which was discussed on page 205, is called the *tangent of the angle*; that is,

$$\text{tangent} = \frac{\text{ordinate}}{\text{abscissa}}.$$

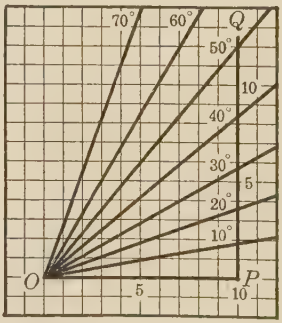


In designating angles it is frequently convenient to use Greek letters. The letters used for this purpose are generally ϕ (*phi*, pronounced fē) and θ (*theta*, pronounced thā'ta).

In the above figure the ordinate of P is QP, or y, and the abscissa of P is OQ, or x. For the tangent of ϕ, which is usually abbreviated in writing to "tan ϕ", we therefore have

$$\tan \phi = \frac{y}{x}.$$

Finding Tangents. If, as shown in the figure below, we lay off angles of various sizes at O on squared paper, and then draw PQ so that $OP = 10$, we can find approximate values for the tangents of the angles by taking $\frac{1}{10}$ of the number of spaces cut off on PQ by the terminal sides of the angles.



For example, we can easily find that

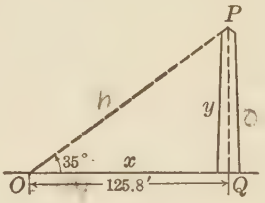
$$\tan 10° = \tfrac{1}{10} \times 1.8 = 0.18,$$
$$\tan 20° = \tfrac{1}{10} \times 3.6 = 0.36,$$
$$\tan 30° = \tfrac{1}{10} \times 5.8 = 0.58,$$

and so on for angles of different sizes.

In practice, however, we find tangents from a table like that given on pages 340 and 341. Thus, if we wish to find tan 10°, we look in the left-hand column on page 340 for **10**, and then at the right in the column headed **0.0°**, or **0′**, we find 0.1763; that is, tan 10° = 0.1763, to the nearest 0.0001.

These tangents are called *natural tangents* in order to distinguish them from the logarithmic tangents which we shall use later.

Using the Tangent. We shall now see how the tangent ratio is used in a practical case. For example, if from a point that is 125.8 ft. from the foot of a monument the angle of elevation of the top is 35°, what is the height of the monument?



Since $\tan \phi = \frac{y}{x}$,

we have $y = x \tan \phi$

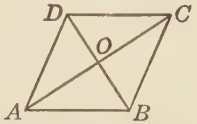
$= 125.8 \tan 35°$.

From the table on page 340 we find that $\tan 35° = 0.7002$, and hence

$$y = 125.8 \times 0.7002 = 88.08516.$$

That is, to the nearest 0.1 ft., the height of the monument is 88.1 ft.

We can also use the table of tangents in finding the size of an angle when we know its tangent. For example, it is proved in geometry that the diagonals of a rhombus (a quadrilateral with equal sides) bisect each other at right angles. If, in the figure below, $BD = 4.2$ in. and $AC = 6.0$ in., what is the size of $\angle OCB$?



From $\tan \phi = \frac{y}{x}$,

we have $\tan OCB = \frac{OB}{OC} = \frac{2.1}{3.0} = 0.7$.

Looking in the table on page 340, we find that the tangent nearest to 0.7, or 0.7000, is 0.7002, which is the tangent of 35°. That is, we thus find that $\angle OCB = 35°$, approximately.

Accuracy of Measurement. In the first example, above, there are five decimal places in the value of y. If all these figures were reliable, this would give us the height to approximately 0.0001 ft., which is obviously absurd. We therefore give the height to the nearest 0.1 ft. because

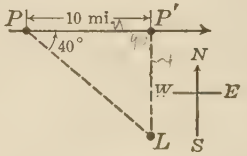
No result can be more nearly accurate than the least accurate of the measurements upon which it is based.

If x had been given as 125 ft., we should need to decide what constitutes a reasonable degree of accuracy. For our purposes we shall assume that it is possible to measure in such cases to the nearest 0.1 ft.

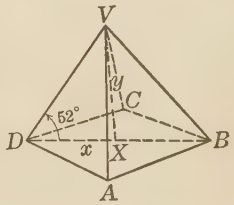
Exercises. Tangents

1. Using squared paper as explained on page 206, lay off with a protractor angles of 12°, 15°, 23°, 28°, 34°, and 40°, and find their tangents to the nearest 0.01 by measurement. Verify the results by the table on page 340.

2. The captain of a ship which is sailing due east observes that a lighthouse L lies 40° south of east when the ship is at the point P. After sailing 10 mi. on the same course to the point P', the lighthouse is found to lie directly south of the ship. Find to the nearest 0.1 mi. the distance from P' to L.



3. This figure represents the Great Pyramid of Egypt, which has been considerably worn away by erosion. From measurements of the edge of the square base it is estimated that the diagonal DB was originally 750 ft. long, and the angle of elevation of the original vertex V from D was approximately 52°. What was the original height of the pyramid?

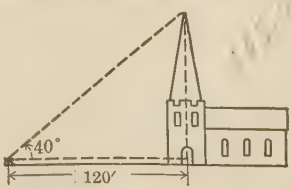


To what degree of accuracy should you give the result in this case?

4. From a point on the ground 38.0 ft. from the foot of a telegraph pole the angle of elevation of the top is found to be 42°. What is the height of the pole?

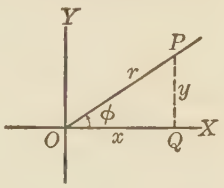
The dimension 38.0 ft. indicates a measurement to the nearest 0.1 ft.

5. With a transit set 120 ft. from the building and 5 ft. above the ground the angle of elevation of the top of a church spire is found to be 40°. How high is the spire?



6. Draw a right triangle in which C is the right angle and $\angle A = 45°$. How can you tell without consulting the table of tangents that $\tan 45° = 1$?

Functions of an Angle. Since the trigonometric ratios depend for their value upon the size of the angle, they are called *functions of the angle*; that is, tan ϕ is a function of ϕ.



In defining other functions of ϕ we shall need to use the line OP, which is called the *distance* of the point P from the origin, and is customarily designated by r.

Besides tan ϕ we shall need to use the following functions of ϕ, the abbreviations for which are shown at the right:

$$\textit{sine of } \phi = \frac{\textit{ordinate}}{\textit{distance}}, \quad \text{or} \quad \sin\phi = \frac{y}{r};$$

$$\textit{cosine of } \phi = \frac{\textit{abscissa}}{\textit{distance}}, \quad \text{or} \quad \cos\phi = \frac{x}{r};$$

$$\textit{cotangent of } \phi = \frac{\textit{abscissa}}{\textit{ordinate}}, \quad \text{or} \quad \cot\phi = \frac{x}{y}.$$

The cotangent, the reciprocal of the tangent, is not so important as the others. It is occasionally used as a multiplier, however, in order to avoid dividing by tan ϕ. The other reciprocal functions, the *secant* (the reciprocal of the cosine) and the *cosecant* (the reciprocal of the sine) can also be used in this way, but we do not really need them.

Since x, y, and r are the sides of a right triangle, we may also define the functions of ϕ as follows:

$$\sin\phi = \frac{\text{opposite side}}{\text{hypotenuse}}, \qquad \tan\phi = \frac{\text{opposite side}}{\text{adjacent side}},$$

$$\cos\phi = \frac{\text{adjacent side}}{\text{hypotenuse}}, \qquad \cot\phi = \frac{\text{adjacent side}}{\text{opposite side}}.$$

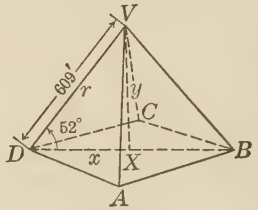
Although we might find an approximate value for any function by the use of squared paper, as explained on page 206, in practical work we always refer to tables in which the values of the functions are conveniently arranged.

In finding the sines and cosines of angles given to the nearest 1° the tables on pages 336–339 are used in the same way as the table of tangents, which was described on page 206.

The use of the table in finding functions when the angles are given to fractions of a degree will be explained later as the need arises.

Using the Sine. The tangent formula involves x and y, and the sine formula involves y and r. The latter is therefore more convenient in cases where it is easier to measure r and ϕ than it is to measure x and ϕ. The following examples illustrate the use of the sine in problems involving practical measurements:

1. Compute the height of the Great Pyramid of Egypt (see Ex. 3, page 208) by taking the length of the edge VD as 609 ft. and the angle which VD makes with the base as approximately 52°.



Since $\sin \phi = \frac{y}{r}$,

we have $y = r \sin \phi$

$= 609 \sin 52°$.

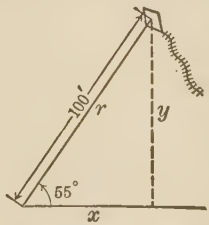
From the table on page 337 we find that

$\sin 52° = 0.7880$;

whence $y = 609 \times 0.7880 = 479.8920$.

We thus find that the height of the pyramid is approximately 480 ft. As suggested in Ex. 3, page 208, the pyramid is so worn away that accurate measurements are not possible.

2. A boy is flying a kite at the end of a 100-foot string which makes an angle of approximately 55° with the ground. Assuming that the string is stretched perfectly straight, how high is the kite above the ground?



As in Ex. 1, above, we have

$y = r \sin \phi$

$= 100 \times \sin 55°$.

From the table on page 337 we find that

$\sin 55° = 0.8192$;

whence $y = 100 \times 0.8192 = 81.92$.

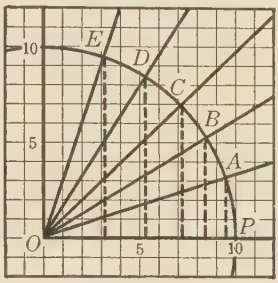
We therefore find that to the nearest 1 ft. the height of the kite is 82 ft.

In deciding upon the degree of accuracy to which it is reasonable to give the answer in this case, we should consider first that the flying kite is always in motion; second, that it is difficult to measure the angle made by the string with the ground; and third, that although we assume that the string is straight, actually it will have considerable sag. An answer to the nearest foot is therefore simply approximate; in fact, it would probably be difficult to measure the actual height within 10 ft.

Exercises. Sines

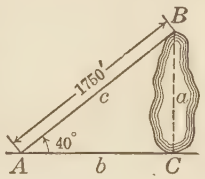
Copy the following table and fill the blank spaces with the approximate values of the sines of the different angles, as determined from the accompanying figure:

	Angle	Size	Sine
1.	*POA*	18°	0.31
2.	*POB*	32°	
3.	*POC*	44°	
4.	*POD*	58°	
5.	*POE*	72°	



6. From the table of sines on pages 336 and 337 find the sine of each angle mentioned in Exs. 1–5.

7. In finding the distance a across the lake shown in this figure, a surveyor laid off the line b at right angles to a. Then taking a point A on b, he measured $\angle A$, finding that it was 40°. He then measured the distance from A to B, which was 1750 ft. What did the surveyor find the distance across the lake to be?



8. Three towns, A, B, and C, are so situated that B, which lies directly north of C, is 34.5 mi. from A in a direction 25° north of east, and C is directly east of A. What is the distance from C to B?

9. A ladder 40 ft. long resting against a building makes an angle of 60° with the ground. How high does the ladder reach on the wall of the building?

10. Find to the nearest 0.01 in., the side of the largest square that can be drawn in a circle $2\frac{1}{2}$ in. in diameter; in one $3\frac{7}{8}$ in. in diameter; in one $5\frac{1}{16}$ in. in diameter.

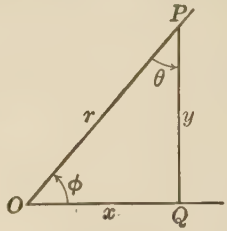
The diagonal of the square is equal to the diameter of the circle, and the diagonal of a square forms angles of 45° with the sides.

Using the Cosine. In the first figure below it is evident from the definition of "sine" that

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{x}{r},$$

a ratio which has already been defined as the cosine of ϕ. That is, since

$$\phi + \theta = 90°,$$



we see that the cosine of ϕ is the sine of the complement of ϕ.

The word "cosine" is thus seen to be merely an abbreviated way of expressing "complement's sine."

There are various practical cases in which it is convenient to use the cosine. For example, suppose that in tearing down a building a contractor wishes to brace the walls of an adjoining building with timbers 24 ft. long which will make an angle of approximately 55° with the horizontal. How far from the wall should the foot of each timber be placed?

Since $\cos\phi = \frac{x}{r}$,

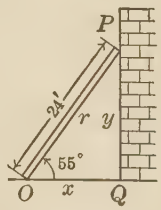
we have $x = r\cos\phi$

$= 24\cos 55°.$

From the table on page 339 we find that

$\cos 55° = 0.5736.$

Then $x = 24 \times 0.5736 = 13.7664.$



That is, the foot of each timber should be placed 13.8 ft., or approximately 13 ft. 10 in., from the wall.

It is evident that we could have solved this problem without using the cosine; that is, we could have used the sine of the complement of $\angle O$ instead.

Thus, $\angle P = 90° - \angle O = 90° - 55° = 35°,$

and $x = r\sin P = 24\sin 35° = 24 \times 0.5736 = 13.7664.$

The result, 13.8 ft., is of course the same as that found above.

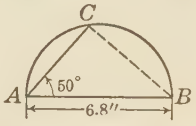
Although we might thus get along without the cosine, its use eliminates the extra step of first computing the complement of the given angle.

Exercises. Cosines

1. Construct an angle whose cosine is 0.5.

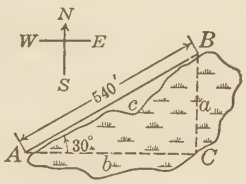
In this case we see that $x = 0.5\ r$, and hence we have $r = 2\ x$.

2. In making a pattern for a carving a workman uses a semicircle 6.8 in. in diameter, drawing AC, as here shown, so that $\angle A = 50°$. Draw a figure to scale and measure the length of AC to the nearest 0.1 in. Then verify the result by trigonometry.



Since it is inscribed in a semicircle, $\angle C = 90°$.

3. A surveyor wishes to measure the distance b on an east-and-west line across a marsh, as shown. Placing his transit at C and sighting northward, he has his helper place a stake at B, from which to measure AB. Moving the transit to A, they find that $\angle A = 30°$ and $AB = 540$ ft. Find the length of b.



4. A man walking southward along a straight road leaves it at a point P and continues along an oblique path which runs straight for a distance of 500 ft. to a spring S. He then takes another path running at right angles to the road and returns to the road at a point Q which is 171 ft. from P. Find the angle at P which the oblique path makes with the road and the angle at S which it makes with the other path.

In speaking of the angle which one line makes with another, we always mean the acute angle unless it is otherwise specified.

5. If an extension ladder 40 ft. long makes an angle of 60° with the ground when the top of the ladder is resting against the side of a building, how far is the foot of the ladder from the wall of the building?

6. When a rope hanging from the top of the 35-foot arm of a derrick touches the ground at a distance of 12 ft. from the foot of the derrick, what approximate angle does the arm make with the ground?

Tables of Natural Functions. We have already learned how to use the four-place tables of natural functions on pages 336–341 for angles given to the nearest 1°. In the nine columns at the right of the column headed **0.0°** (or **0′**) are the functions for every 0.1°, or 6′, from 0° to 90°.

Thus, to find sin 30° 48′, we look on page 336 for **30** in the column of degrees at the left and on this line, in the column headed **48′** (or **0.8°**), we find .5120; that is, sin 30° 48′ = 0.5120.

The methods of using the difference columns at the right of each page, or of using ordinary interpolation when necessary, are much the same as have been already described for the table of logarithms (see page 72). The following examples will illustrate the methods used in such cases:

1. Find sin 20° 39′.

From the table on page 336 we find that sin 20° 36′ = 0.3518. Since 20° 39′ is 3′ greater than 20° 36′, we follow along horizontal line **20** to the difference column headed **3′**, and find 8; that is, the difference for 3′ is 0.0008. We therefore have

$$\sin 20° 39' = 0.3518 + 0.0008 = 0.3526.$$

2. Find tan 80° 44′.

We see on page 341 that 80° 44′ lies between 80° 42′ and 80° 48′ and that the heavy-type figures of tan 80° 36′ change the integral part 5 of tan 80° 0′ to 6 for the rest of the line for 80°. Hence we have

$$\begin{aligned} \tan 80° 48' &= 6.1742 \\ \tan 80° 42' &= \underline{6.1066} \end{aligned}$$

Subtracting, the difference for 6′ is 0.0676

The difference for 2′ is then $\frac{2}{6} \times 0.0676$, or 0.0225. We therefore have

$$\tan 80° 44' = 6.1066 + 0.0225 = 6.1291.$$

3. Find the acute angle for which the cosine is 0.3377.

In the table on page 339 we find that 0.3377 lies between 0.3387 and 0.3371, the cosines of 70° 12′ and of 70° 18′, which have a total difference of 0.0016. Since 0.3377 is 0.0010 less than 0.3387, we see that this difference corresponds to $\frac{10}{16}$ of 6′, or 4′, to the nearest 1′. Then since the cosine decreases as the angle increases from 0° to 90°, we see that the required angle is 4′ greater than 70° 12′; that is, it is 70° 16′.

Tables of Logarithmic Functions. Computations involving the trigonometric functions can often be simplified by using logarithms. For example, in a case like

$$r = \frac{y}{\sin \phi} = \frac{216.5}{0.7490},$$

it is apparent that a difficult division can be avoided. Instead of having to look up the value of sin ϕ in one table and the logarithm of this value in another, it will evidently be much more convenient if we use a table which gives the logarithms of the sines, or the *logarithmic sines*, as they are called.

On pages 342–347 there are given four-place tables of the logarithmic sines, cosines, and tangents for each 0.1°, or 6′, from 0° to 90°. The method of using these tables of logarithmic functions is simply a combination of the methods which we have already learned for the tables of natural functions and the table of four-place logarithms.

In the tables the logarithms of the functions are printed in the customary form $\bar{1}.2397$, which may, of course, be written as $9.2397 - 10$, as $0.2397 - 1$, or in any other way that gives the characteristic in the form most convenient for computation.

When very small angles or angles near 90° are used, it is necessary to consult special tables. We shall not, however, need to use such angles in our work.

The student should verify the following examples by the tables in order to be sure that the method of finding the logarithmic function in each case is perfectly clear. It should be understood that the indicated additions and subtractions are usually performed mentally.

(1) From page 342, $\log \sin 20° 34' = \bar{1}.5443 + .0014 = \bar{1}.5457$.
(2) From page 345, $\log \cos 84° 45' = \bar{2}.9655 - .0039 = \bar{2}.9616$.
(3) From page 347, $\log \tan 88° 16' = 1.5027 + .0165 = 1.5192$.
(4) From page 343, $\text{antilog sin } \bar{1}.9743 = 70° 30'$, or 70.5°.
(5) From page 344, $\text{antilog cos } \bar{1}.9620 = 23° 36' + 2' = 23° 38'$.
(6) From page 347, $\text{antilog tan } 0.3054 = 63° 36' + 4' = 63° 40'$.

In the third and fifth cases it will be found necessary to use ordinary interpolation, which is applied to these tables in the same way as it is to the tables of natural functions.

Exercises. Use of the Tables

From the tables of natural functions find

1. $\sin 38.6°$.
2. $\cos 39.5°$.
3. $\tan 72.8°$.
4. $\cos 55° 18'$.
5. $\tan 16° 24'$.
6. $\sin 48° 48'$.
7. $\sin 28° 22'$.
8. $\tan 18° 8'$.
9. $\cos 56° 45'$.
10. $\tan 22° 57'$.
11. $\cos 76° 35'$.
12. $\sin 13° 28'$.
13. $\sin 27.52°$.
14. $\cos 19.68°$.
15. $\tan 77.65°$.
16. $\sin 60° 16'$.
17. $\tan 84° 19'$.
18. $\cos 86° 15'$.

From the tables find the value of ϕ when

19. $\sin \phi = 0.1685$.
20. $\cos \phi = 0.8590$.
21. $\sin \phi = 0.1694$.
22. $\cos \phi = 0.8587$.
23. $\tan \phi = 3.0287$.
24. $\cos \phi = 0.9708$.
25. $\sin \phi = 0.9873$.
26. $\tan \phi = 0.2384$.
27. $\cos \phi = 0.9950$.

From the tables of logarithmic functions find

28. log sin 15.2°.
29. log tan 72.1°.
30. log cos 38.8°.
31. log sin 2° 6′.
32. log cos 9° 12′.
33. log tan 5° 30′.
34. log cos 18° 22′.
35. log sin 44° 15′.
36. log tan 16° 57′.
37. log sin 65° 46′.
38. log cos 28° 17′.
39. log tan 84° 25′.
40. log sin 35.16°.
41. log tan 27.82°.
42. log cos 45.25°.
43. log tan 60° 33′.
44. log cos 27° 27′.
45. log tan 82° 15′.

From the tables find the value of ϕ when

46. $\log \sin \phi = \bar{1}.9435$.
47. $\log \cos \phi = \bar{3}.8439$.
48. $\log \tan \phi = 0.5644$.
49. $\log \cos \phi = \bar{1}.8801$.
50. $\log \tan \phi = 0.0816$.
51. $\log \sin \phi = \bar{2}.9088$.
52. $\log \tan \phi = \bar{1}.1078$.
53. $\log \cos \phi = \bar{1}.5371$.

54. Find $\sin 34° 52'$ by using the difference columns and also by ordinary interpolation. By how much do the results differ? Will this affect the result in finding $20.1 \sin 34° 52'$?

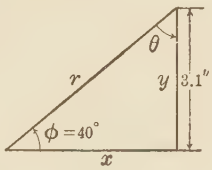
Solving the Right Triangle. The formula for the sine involves ϕ, y, and r; that for the cosine, ϕ, x, and r; and that for the tangent, ϕ, y, and x. Since $\phi + \theta = 90°$, we can therefore solve a right triangle completely if we know

1. One acute angle (ϕ or θ) and one side (x, y, or r);
2. Two sides (x and y, x and r, or y and r).

The following examples illustrate the method of procedure:

1. Solve the triangle in which $\phi = 40°$ and $y = 3.1$ in.

Drawing the figure, as shown at the right, we may proceed as follows, arranging the work in columns for convenience in checking:



(1) $\theta = 90° - \phi$
$= 90° - 40°$
$= 50°$.

(2) From $\tan \phi = \frac{y}{x}$,

we have $x = \frac{y}{\tan \phi} = \frac{3.1}{\tan 40°}$.

$\log 3.1 = 0.4914$
$\log \tan 40° = \bar{1}.9238$
$\log x = 0.5676$

Hence $x = 3.695$,
or $x = 3.7$ in.

(3) From $\sin \phi = \frac{y}{r}$,

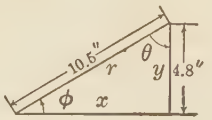
we have $r = \frac{y}{\sin \phi} = \frac{3.1}{\sin 40°}$.

$\log 3.1 = 0.4914$
$\log \sin 40° = \bar{1}.8081$
$\log r = 0.6833$

Hence $r = 4.823$,
or $r = 4.8$ in.

2. Solve the triangle in which $y = 4.8$ in. and $r = 10.5$ in.

First drawing the figure, we have



(1) $\sin \phi = \frac{y}{r}$
$= \frac{4.8}{10.5}$.

$\log 4.8 = 0.6812$
$\log 10.5 = \underline{1.0212}$
$\log \sin \phi = \bar{1}.6600$

Hence $\phi = 27° 12'$.

(2) $\theta = 90° - \phi$
$= 90° - 27° 12'$
$= 62° 48'$.

(3) From $\cos \phi = \frac{x}{r}$, we have

$x = r \cos \phi = 10.5 \cos 27° 12'$.

$\log 10.5 = 1.0212$
$\log \cos 27° 12' = \bar{1}.9491$
$\log x = 0.9703$

Hence $x = 9.340$,
or $x = 9.3$ in.

Exercises. Solving the Right Triangle

Solve the right triangle with the hypotenuse r, the sides x and y, and the acute angles ϕ and θ when

1. $\phi = 41° 35'$, $y = 11.2$ in.
2. $\phi = 35° 16'$, $x = 28.5$ ft.
3. $\phi = 17° 28'$, $r = 64.1$ ft.
4. $\theta = 51° 9'$, $r = 125.7$ mi.
5. $\theta = 13° 23'$, $y = 22.9$ cm.
6. $\theta = 59° 10'$, $x = 17.5$ m.
7. $\phi = 35° 38'$, $r = 62.8$ ft.
8. $\theta = 22° 22'$, $x = 22.2$ in.
9. $y = 8.2$ ft., $x = 19.5$ ft.
10. $y = 7.0$ mi., $r = 14.2$ mi.
11. $r = 7.14$ in., $x = 6.38$ in.
12. $x = 23.7$ cm., $y = 9.7$ cm.
13. $r = 14.0$ in., $y = 10.3$ in.
14. $y = 29.8$ m., $x = 17.4$ m.
15. $x = 19.4$ ft., $y = 14.2$ ft.
16. $r = 7.46$ mi., $x = 5.67$ mi.

17. A flagpole is broken and blown over by the wind, the two parts remaining joined. If the top of the pole touches the ground at a point 13.5 ft. from the foot and makes an angle of 60° 25′ with the ground, what are the respective lengths of the two parts into which the pole is broken? What was the original height of the pole?

18. In order to find the width of a river a surveying party laid off a straight line 450.0 ft. long from A to C along one bank. One member of the party stood at C and located a point B on the opposite bank on a line at right angles to AC. Another member, standing at A, measured $\angle CAB$ and found that it was 59.6°. How wide was the river at C?

19. The sloping face of a railroad embankment is 23.5 ft. long, and the top of the embankment is 19.6 ft. above the surrounding level. At what angle does the sloping face of the embankment meet the ground?

20. In flying from a city A to a city B, which is 275.2 mi. to the east and 106.5 mi. to the north of A, what course should an aviator follow?

The course is described by the angle between it and the north-and-south line through A.

21. A rectangular field 850.0 ft. long and 520.0 ft. wide is divided into two equal parts by a straight fence that starts at a point on one of the long sides 110.0 ft. from a corner. How long is the fence which divides the field?

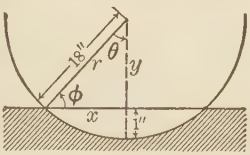
22. From a point 30 ft. above the water, on the deck of a steamer, the angle of elevation of the light in a lighthouse which is 105 ft. high is found to be 0° 52′. How far is the steamer from the lighthouse?

23. What is the gradient (the angle of slope) of a road which rises 1.0 ft. in every 75.0 ft. measured horizontally? 1.0 ft. in every 75.0 ft. measured along the road?

24. From a point A a railroad runs upward at a uniform gradient of 8° 40′ to a point B, which is 1430.0 ft. from A, measured horizontally. From B the road runs at a zero gradient (that is, horizontally) for a distance of 1675.0 ft. to C. What would have been the gradient if there had been a uniform rise from A to C?

25. A parallelogram has adjacent sides of 12.7 in. and 10.8 in., and the angle included between these sides is 80° 15′. Find the area of the parallelogram.

26. What is the approximate area of the bearing surface when a cylindric roller which is 6 ft. long and 36 in. in diameter makes a depression in the soil of 1 in.? of 2 in.?

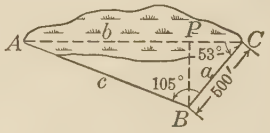


The required area evidently bears the same relation to the entire area of the curve surface of the roller that the number of degrees in 2θ bears to 360. What parts of a right triangle are given? How can we find the size of θ?

27. If two pulleys, which are respectively 6 ft. and 2 ft. in diameter, are 20 ft. apart between centers, what length of belting running straight between the pulleys is needed to connect them? What is the length if the belt is crossed?

The belt will be tangent to each pulley, and a tangent to a circle is always perpendicular to the radius drawn to the point of contact.

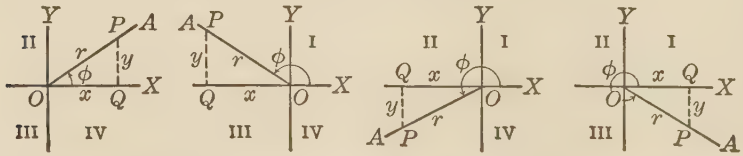
Oblique Triangles. Each triangle that we have considered thus far has contained a right angle. In practice, however, we often find that the triangle with which our measurements are concerned contains only oblique angles.



For example, this figure shows how a surveyor measured the length AC of a swamp. On account of trees and other obstructions he could not lay off a perpendicular to AC at either end, and so he ran the line CB for a distance of 500 ft. at an angle of 53° with AC. He then found that the angle between the lines AB and BC was 105°.

We might find the length of AC by drawing the altitude BP, thus dividing $\triangle ABC$ into two right triangles, but there is a more convenient method. Since $\angle B$ is obtuse, we must, however, first learn how to find the functions of obtuse angles.

Angles in any Quadrant. On pages 206 and 209 we defined $\sin\phi$, $\cos\phi$, $\tan\phi$, and $\cot\phi$ in terms of the abscissa x, the ordinate y, and the distance r, using for this purpose a figure in which the terminal side of ϕ was in the first quadrant.



If, as shown in the above figures, we allow ϕ to increase, its terminal side lying in each of the four quadrants in turn, we still apply the same definitions; that is, we have

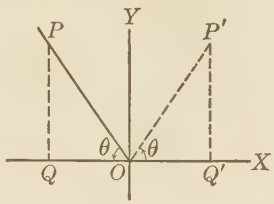
$$\sin\phi=\frac{y}{r},\quad \cos\phi=\frac{x}{r},\quad \tan\phi=\frac{y}{x},\quad \cot\phi=\frac{x}{y}.$$

The distance r is always positive, but since the signs of x and y change with the quadrants, it is evident that some of the functions will be negative.

Since we shall need to consider only obtuse angles, we may disregard all angles in the third and fourth quadrants.

Functions of an Obtuse Angle. If in the figure below we let $\angle XOP$ represent any obtuse angle, and if we then designate $\angle POQ$, its supplement, by θ, it is evident that we may write

$$\angle XOP = 180° - \theta.$$



Constructing $\angle XOP'$ in the first quadrant equal to θ and taking OP' equal to OP, we see that $\triangle OPQ$ is congruent to $\triangle OP'Q'$, and hence it follows that $OQ = OQ'$ and $QP = Q'P'$.

Then $$\sin (180° - \theta) = \frac{QP}{OP} = \frac{Q'P'}{OP'} = \sin \theta.$$

Similarly, $$\cos (180° - \theta) = \frac{OQ}{OP} = -\frac{OQ'}{OP'} = -\cos \theta,$$

the negative sign being used because OQ' is opposite in direction to OQ. Treating the tangent and the cotangent in the same way, we thus have the following formulas:

$\sin (180° - \theta) = \sin \theta$ $\qquad$ $\tan (180° - \theta) = -\tan \theta$

$\cos (180° - \theta) = -\cos \theta$ $\qquad$ $\cot (180° - \theta) = -\cot \theta$

That is, the functions of an obtuse angle are numerically equal to the corresponding functions of its supplement, and the cosine, tangent, and cotangent are always negative.

Thus, $\sin 113° = \sin (180° - 113°) = \sin 67° = 0.9205,$

and $\tan 113° = -\tan (180° - 113°) = -\tan 67° = -2.3559.$

Exercises. Functions of Obtuse Angles

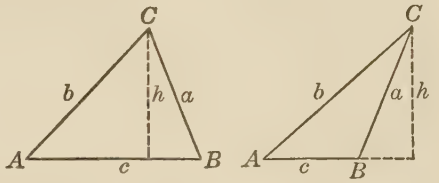
From the tables of natural functions find

1. $\sin 117°$.
2. $\cos 125°$.
3. $\tan 150°$.
4. $\tan 163° 15'$.
5. $\sin 154° 30'$.
6. $\cos 135° 0'$.
7. $\cos 95° 5'$.
8. $\sin 98° 10'$.
9. $\tan 108° 10'$.

10. Show that the sine of an obtuse angle ϕ is numerically equal to $\cos (\phi - 90°)$ and that $\cos \phi = -\sin (\phi - 90°)$.

Sine Law. If in each of the two oblique triangles shown below we draw the altitude h from the vertex C, we see that in both cases we have

$$\frac{h}{b} = \sin A.$$



In the first case, also,

$$\frac{h}{a} = \sin B,$$

and in the second figure, where h is outside $\triangle ABC$, we have

$$\frac{h}{a} = \sin (180° - B) = \sin B.$$

If we now divide $\frac{h}{b} = \sin A$ by $\frac{h}{a} = \sin B$, member for member, we have

$$\frac{a}{b} = \frac{\sin A}{\sin B},$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

By drawing the altitude from the vertex A, it can then be shown in the same way that

$$\frac{b}{c} = \frac{\sin B}{\sin C}, \quad \text{or} \quad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

From these proportions we have the following important relation, which is called the *Sine Law*:

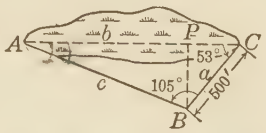
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This law, which involves the three sides and the three angles, may be expressed in words as follows:

In any triangle the ratios of the sides to the sines of the opposite angles are equal.

On the following page we shall see that the use of this law in solving an oblique triangle makes it unnecessary first to divide the triangle into two right triangles.

Sine Law Applied. In order to illustrate the advantages of the Sine Law we shall now apply it to the case discussed on page 220. Stated as a problem, we are required to find the side AC, or b, of an oblique triangle in which $\angle B = 105°$, $\angle C = 53°$, and the side BC, or a, measures 500 ft.



Since we are given $\angle B$, $\angle C$, and a, it is evident that we must first find $\angle A$ in order to use the ratios involving a, b, $\sin A$, and $\sin B$ in the Sine Law. The sum of the angles of any triangle is 180°, and hence we first have

$$A = 180° - (B + C) = 180° - 158° = 22°.$$

From
$$\frac{a}{\sin A} = \frac{b}{\sin B},$$

we then have
$$b = \frac{a \sin B}{\sin A}$$
$$= \frac{500 \sin 105°}{\sin 22°}.$$

Performing this computation by means of logarithms, we have

$$\log 500 = 2.6990$$
$$\log \sin 105° = \log \sin 75° = \bar{1}.9849$$
$$\operatorname{colog} \sin 22° = 0.4264$$
$$\log b = 3.1103$$

Then
$$\text{antilog } 3.1103 = 1289;$$

that is, the length of the swamp to four figures, which is as close as we can find it with a four-place table, is 1289 ft.

The Sine Law was developed by dividing an oblique triangle into two right triangles. The solution just given thus substitutes one solution for two.

That is, after drawing the altitude BP, we might first find that

$$BP = a \sin C = 500 \sin 53° = 500 \times 0.7986 = 399.3,$$

and
$$PC = a \cos C = 500 \cos 53° = 500 \times 0.6018 = 300.9.$$

Then, after finding, as above, that $\angle A = 22°$, we would have

$$AP = \frac{BP}{\tan A} = \frac{399.3}{0.4040} = 988.4.$$

Hence
$$b = AP + PC = 988.4 \text{ ft.} + 300.9 \text{ ft.} = 1289.3 \text{ ft.}$$

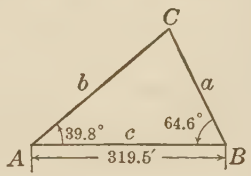
Although the two results agree, it is evident that the first solution is much the easier.

Exercises. The Sine Law

Solve the oblique triangles in which

1. $A = 32°$, $B = 73°$, and $a = 8.6$ in.
2. $B = 41°$, $C = 86°$, and $b = 47$ ft.
3. $A = 38.4°$, $B = 49.9°$, and $a = 72.3$ cm.
4. $B = 79.8°$, $C = 82.6°$, and $b = 398.2$ mi.
5. $C = 76.7°$, $A = 83.4°$, and $c = 426.8$ yd.
6. $A = 49°\ 7'$, $B = 89°\ 9'$, and $a = 12.04$ ft.

7. In mapping the triangular area here shown, the surveyors first took a base line AB that was 319.5 ft. long. They then found by measuring that $\angle A = 39.8°$ and that $\angle B = 64.6°$. How far is C from both A and B?



In mapping small areas, the curvature of the earth's surface is disregarded; that is, the area is considered to lie in a horizontal plane.

8. Check the result of Ex. 7 by showing that $b \sin A = a \sin B$.

9. At a point on the bank of a river the angle of elevation of the top of a tree on the opposite bank is 59° 43′, and from a point 42 ft. farther off in a straight line from the base of the tree the angle of elevation is 49° 50′. What is the width of the river?

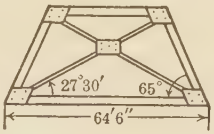
10. At a point on a uniform slope of 8° 53′ the angle of elevation of the top of a tree growing vertically is 42° 17′, and 75 ft. farther down the slope in a straight line from the tree the angle is 21° 47′. Find the height of the tree.

11. A cylindric water tank stands on a slope which rises vertically 20 ft. for every 100 ft. measured horizontally. From a point on the slope below the foot of the tank the angle of elevation of the top is 43° 18′, while from a point 100 ft. farther down the slope, measured horizontally, the angle of elevation is 27° 15′. How high is the tank?

12. The front wall of a building that stands on a side hill makes an angle of 108° 16′ with the ground. From a point 92 ft. downhill from the building, measured along the slope, the line of sight to the top makes an angle of 24° 10′ with the slope of the hill. What is the height of the building?

13. From one end of a base line 1250 ft. long the line of sight to a distant landmark makes an angle of 36° 28′ with the base line, and from the other end the angle is 112° 32′. How far is the landmark from each end of the base line?

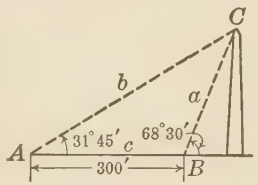
14. In a bridge truss which has the shape shown in this figure the sloping ends make an angle of 65° with the base, and the diagonal braces make an angle of 27° 30′ with the base. Find the lengths of the ends, the top, and the diagonals of a truss 64 ft. 6 in. long.



15. From a point at the foot of a mountain the angle of elevation of the top is 43° 20′. From a point $\frac{1}{2}$ mi. up a trail that rises at a uniform slope of 32° the angle of elevation of the top is 49° 38′. Find the height of the mountain in feet.

The angle of elevation in each case is measured from the horizontal.

16. From a point A the angle of elevation of the top of a monument is found to be 31° 45′, while from the point B, which is 300 ft. nearer the base of the monument in a straight line, the angle of elevation of the top is found to be 68° 30′. Find the height of the monument.



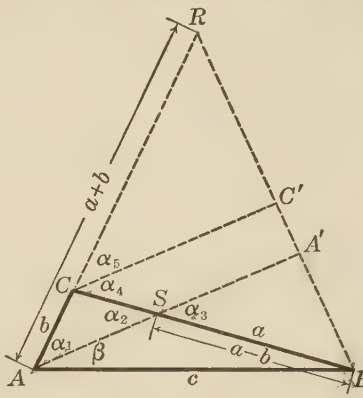
17. From Ex. 16 develop a formula for finding the height h of an object in terms of the angles of elevation α and β taken from two points in line with the object and at the distance d from each other.

18. Make up a problem similar to Ex. 16 and solve it by applying the formula developed in Ex. 17.

The Tangent Law. Another relation that is convenient in solving the oblique triangle is called the *Tangent Law.* In developing this law we may take an oblique triangle, like $\triangle ABC$ here shown, and produce the side b to R until $AR = a + b$. If on a we then lay off CS equal to b, we have $SB = a - b$. If through S we now draw AS, producing it to cut BR at A', and if through C we draw CC' parallel to AA', and letter the angles as shown in the figure, we have

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4,$$

and $\alpha_5 = \alpha_1 = \alpha_4$.



Since CC' bisects $\angle BCR$ and $CR = CB$, CC' is perpendicular to BR, and so is its parallel, AA'. Hence $\triangle SBA'$ and ARA' are similar, because they are right triangles in which $\alpha_3 = \alpha_1$.

Now $\angle BCR$ is an exterior angle of $\triangle ABC$, and hence we have

$$\alpha_1 = \alpha_5 = \tfrac{1}{2} \angle BCR = \tfrac{1}{2}(A + B);$$

whence

$$\beta = A - \alpha_1 = A - \tfrac{1}{2}(A + B) = \tfrac{1}{2}(A - B).$$

Then

$$\tan \tfrac{1}{2}(A - B) = \tan \beta = \frac{BA'}{AA'},$$

and

$$\tan \tfrac{1}{2}(A + B) = \tan \alpha_1 = \frac{RA'}{AA'};$$

whence

$$\frac{\tan \tfrac{1}{2}(A - B)}{\tan \tfrac{1}{2}(A + B)} = \frac{BA'}{RA'}.$$

Now

$$\frac{BA'}{RA'} = \frac{BS}{RA} = \frac{a - b}{a + b},$$

and hence

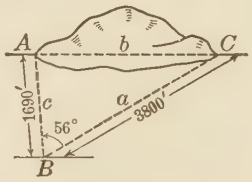
$$\frac{\tan \tfrac{1}{2}(A - B)}{\tan \tfrac{1}{2}(A + B)} = \frac{a - b}{a + b}.$$

Similar relations involving B, C, b, and c, or A, C, a, and c are easily developed in the same way, thus giving us the three formulas which state the Tangent Law.

The Tangent Law Applied. Expressed in words, the Tangent Law developed on page 226 states that

In any triangle the tangent of half the difference between two angles is to the tangent of half their sum as the difference between the two opposite sides is to the sum of these sides.

To illustrate the application of this law, suppose that in finding the straight-line distance from A to C on opposite sides of a hill a surveyor sets up his transit at B, as here shown. He finds that $\angle B = 56°$, that $a = 3800$ ft., and that $c = 1690$ ft. What is the distance from A to C?



Since $\angle B$ is the angle given, it is evident that we must use the tangent formula involving $\measuredangle A$ and C. We therefore proceed as follows:

$$A + C = 180° - B = 180° - 56° = 124°;$$

whence $$\tfrac{1}{2}(A + C) = 62°$$

Also, $$a - c = 3800 - 1690 = 2110,$$

and $$a + c = 3800 + 1690 = 5490.$$

Substituting these three values in the formula

$$\frac{\tan \tfrac{1}{2}(A - C)}{\tan \tfrac{1}{2}(A + C)} = \frac{a - c}{a + c},$$

the Tangent Law for A and C, we have

$$\frac{\tan \tfrac{1}{2}(A - C)}{\tan 62°} = \frac{2110}{5490},$$

or $$\tan \tfrac{1}{2}(A - C) = \tfrac{2}{5}\tfrac{1}{4}\tfrac{1}{9}\tfrac{0}{0} \tan 62°.$$

Now
$$\log 2110 = 3.3243$$
$$\log \tan 62° = 0.2743$$
$$\operatorname{colog} 5490 = \overline{4}.2604$$
$$\log \tan \tfrac{1}{2}(A - C) = \overline{1}.8590$$

Hence $\tfrac{1}{2}(A - C) = 35° 51'$.

From $\tfrac{1}{2}(A + C) = 62°$

and $\tfrac{1}{2}(A - C) = 35° 51'$,

we have $A = 97° 51'$,

and $C = 26° 9'$.

From page 222 we then have

$$b = \frac{c \sin B}{\sin C}$$

$$= \frac{1690 \sin 56°}{\sin 26° 9'}.$$

Now
$$\log 1690 = 3.2279$$
$$\log \sin 56° = \overline{1}.9186$$
$$\operatorname{colog} \sin 26° 9' = 0.3558$$
$$\log b = 3.5023$$

Hence $b = 3179$.

That is, to four figures, which is as accurately as we can find b when using a four-place table, the distance from A to C is 3179 ft.

In this case it should be observed that the given parts of the triangle are two sides and the included angle, while on page 223 the given parts were two angles and the included side.

Exercises. The Tangent Law

Solve the oblique triangles in which

1. $a = 23$, $b = 15$, $C = 80°$. **3.** $a = 97.7$, $c = 94.0$, $B = 70° 12'$.

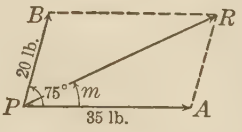
2. $a = 48$, $b = 20$, $C = 60°$. **4.** $b = 57.8$, $c = 74.8$, $A = 28° 10'$.

5. If the lengths of two sides of a triangular field are respectively 20 rd. and 30 rd. and the included angle is 45°, what is the length of the third side?

6. Two straight roads which intersect at an angle of 60° at a point P lead to two towns that are respectively 10.0 mi. and 16.2 mi. from P. How far apart are the towns?

7. From a point C the distance to a tree A on one side of a knoll is 2640 ft., and the distance to a tree B on the opposite side of the knoll is 2270 ft. If the angle included between CA and CB is 73° 15′, what is the distance between the trees?

8. In this figure, PA and PB are drawn to scale to represent forces of 35 lb. and 20 lb. respectively acting from the point P at an angle of 75° with each other. It is shown in physics that the diagonal PR of the parallelogram of which PA and PB are adjacent sides represents the magnitude and direction of the resultant; that is, of the single force which has the same effect as the two given forces. Find the magnitude and direction (the angle marked m) of the resultant PR.



9. If two forces of 15 lb. and 30 lb. respectively are acting from the same point at an angle of 50° with each other, what is the magnitude of the resultant? What angle does the resultant make with the larger force?

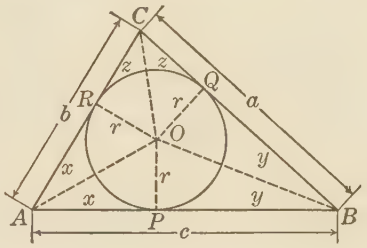
10. If a man is traveling due north in an automobile at the rate of 25 mi./hr. while the wind is blowing from the northwest at the rate of 30 mi./hr., from what direction and at what rate does the wind seem to the driver to blow?

Tangent of Half an Angle. A third convenient relation in solving an oblique triangle involves the use of formulas for the tangent of half an angle. In developing these formulas we shall need to make use of the geometric fact that the center of the inscribed circle is the intersection of the bisectors of the three angles. We shall also need to use the formula

$$T = \sqrt{s(s-a)(s-b)(s-c)},$$

which, as is proved in geometry, gives the area T of a triangle in terms of a, b, c, the sides, and s, the semiperimeter; that is,

$$s = \tfrac{1}{2}(a+b+c).$$



In the figure given above, if we designate the radius of the inscribed circle by r and the pairs of equal tangents by x, y, and z, as shown, for the area T of the triangle we have

$$T = \tfrac{1}{2}\, r(a+b+c) = rs,$$

and hence

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Also,

$$x + y + z = \tfrac{1}{2}(a+b+c) = s;$$

whence

$$x = s - (y+z) = s - a.$$

Now

$$\tan \tfrac{1}{2} A = \frac{r}{x},$$

and hence

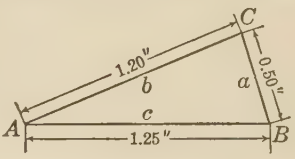
$$\tan \tfrac{1}{2} A = \frac{r}{s-a}.$$

In the same way, $y = s - b$ and $z = s - c$, and hence the corresponding formulas for $\tan \frac{1}{2} B$ and $\tan \frac{1}{2} C$ are

$$\tan \tfrac{1}{2} B = \frac{r}{s-b} \quad \text{and} \quad \tan \tfrac{1}{2} C = \frac{r}{s-c}.$$

We thus have formulas for finding the three angles of a triangle when the three sides are given. In the formulas for the angles it should be observed that the denominator always contains the side opposite the angle, while the formula for r involves all three of the sides.

Half-Angle Formulas Applied. For example, suppose that we wish to find the size of the angle included between the two long sides of a triangular refracting prism in which the lengths of the three sides, as shown in the accompanying figure, are 0.50 in., 1.20 in., and 1.25 in. respectively.



We have $s = \frac{1}{2}(a + b + c) = \frac{1}{2}(0.50 + 1.20 + 1.25) = 1.475$;

whence $s - a = 0.975$, $s - b = 0.275$, and $s - c = 0.225$.

Then $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\frac{0.975 \times 0.275 \times 0.225}{1.475}}$$

$\log 0.975 = \bar{1}.9890$
$\log 0.275 = \bar{1}.4393$
$\log 0.225 = \bar{1}.3522$
$\operatorname{colog} 1.475 = \bar{1}.8312$
$\log r^2 = \bar{2}.6117$

Dividing $\bar{2}.6117$ by 2, we have
$\log r = \bar{1}.3059$.

Now $\tan \frac{1}{2} A = \frac{r}{s-a}$.

$\log r = \bar{1}.3059$
$\log (s-a) = \bar{1}.9890$
$\log \tan \frac{1}{2} A = \bar{1}.3169$

Hence $\frac{1}{2} A = 11° 43'$,
and $A = 23° 26'$.

Exercises. Formulas for Half Angles

Find the sizes of the three angles of a triangle in which

1. $a = 1.6$, $b = 1.9$, $c = 2.1$.
2. $a = 2.5$, $b = 3.7$, $c = 4.2$.
3. $a = 8.3$, $b = 9.8$, $c = 7.6$.
4. $a = 2.9$, $b = 1.1$, $c = 3.5$.

5. Three towns, A, B, and C, are so situated that A is 200 mi. from B and 184 mi. from C, while B is 150 mi. due north from C. How far is A north of C?

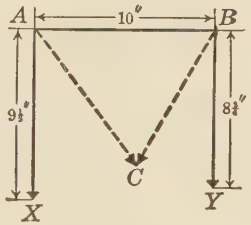
6. Two targets 3 mi. apart are respectively 3068 yd. and 3479 yd. from a fort. When a gun is pointed at one target, through what angle must it be turned to fire on the other?

7. A farmer has a four-sided field $ABCD$ in which the lengths of the sides AB, BC, CD, and DA are respectively 80 rd., 82 rd., 90 rd., and 77 rd., while the diagonal AC is 115 rd. long. What are the sizes of the angles at the four corners?

Exercises. Review

1. When a ship, which is sailing on the course AB, is at A, the line of sight to a lighthouse C makes an angle of 27° 30′ with AB. After sailing 8.2 mi. to B, the angle to the lighthouse becomes 116° 15′. How far is the ship from the lighthouse at each observation?

2. Two pendulums, one $9\frac{1}{2}$ in. long and the other $8\frac{3}{4}$ in., are hung from points 10 in. apart. Through how many degrees of arc must each pendulum swing so that their ends shall just touch each other?



The required angle for the first pendulum is evidently twice $\angle XAC$, and to find this angle we must first find $\angle CAB$.

3. While flying in a fog over a city A towards a city C, which is 54 mi. from A, an aviator loses his way. When the fog lifts, he finds himself over a city B, which is 87 mi. from A and 38 mi. from C. How many degrees must he change his course in order to reach C?

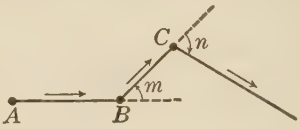
4. In response to an S O S call from a ship at A, another ship at B, 175 mi. due east of A, starts toward A at a speed of 12 mi./hr. At the same time a third ship at C, which is 186 mi. from B in a direction bearing 3° 15′ west of north, also starts for A at a speed of 16 mi./hr. Which ship will reach A first, and how long will it take?

5. Find the length of the diagonal of a regular pentagon that is 5.24 in. on a side.

6. From an east-and-west automobile highway a branch highway bears 22° 30′ west of north. At a point 5.6 mi. east of the road fork a dirt road runs through the woods to a point on the branch highway 4.8 mi. from the fork. How many miles could a motorist save by taking the dirt road instead of following the main road through the fork?

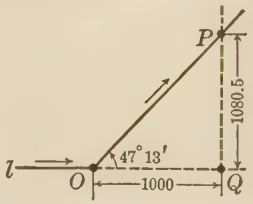
Plotting Angles. In plotting a survey it is necessary to be able to construct angles with great accuracy. With the ordinary protractor, which is marked in degrees and half degrees, it is evident that the construction of an angle like 47° 13′ can be only an approximation. By using the trigonometric functions, however, we can plot angles with a high degree of accuracy.

In designating angles in surveying, the words "right" and "left" are often used to indicate the direction in which the angle is turned. Thus, in this figure, m is designated as an *angle left* and n as an *angle right*, the rotation being referred to the preceding straight line AB produced in the direction in which the survey is proceeding, as indicated by the arrows.



Tangent Method. There are two common methods of plotting angles by means of the trigonometric functions. The first of these is known as the *tangent method*.

For example, suppose that at the point O on the line l shown in this figure we wish to plot an angle left of 47° 13′.



From the table on page 341 we find that

$$\tan 47° 13' = 1.0805.$$

Now $$\text{tangent} = \frac{\text{ordinate}}{\text{abscissa}},$$

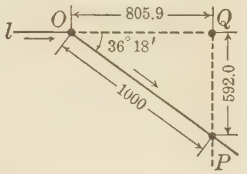
and hence if we assume an abscissa of 1000 units, we shall have

$$\tan O = \frac{1080.5}{1000}.$$

That is, we may first lay off OQ as 1000 units, construct a perpendicular to l at Q, and on this perpendicular lay off QP as 1080.5 units. If we then draw OP, the $\angle QOP$ will contain 47° 13′, as required.

The scale used will, of course, depend upon the circumstances. Thus, in some cases a convenient scale will be 1 in. = 100 units, in others it will be 1 in. = 200 units, and so on.

Sine-Cosine Method. To illustrate the second method, which is called the *sine-cosine method* of plotting an angle, suppose that at O on the line l here shown we wish to plot an angle right of 36° 18′.



Since $$\text{sine} = \frac{\text{ordinate}}{\text{distance}}$$

and $$\text{cosine} = \frac{\text{abscissa}}{\text{distance}},$$

it is evident that if we assume a distance of 1000 units, we can find the abscissa and ordinate from the tables. That is,

$$\sin 36° 18' = 0.5920 \quad \text{and} \quad \cos 36° 18' = 0.8059,$$

and hence $$\sin O = \frac{592.0}{1000} \quad \text{and} \quad \cos O = \frac{805.9}{1000}.$$

We therefore lay off OQ as 805.9 units, construct a perpendicular to l at Q, and on it lay off QP as 592.0 units. Then $\angle POQ$ is the required angle of 36° 18′.

In this method the work is easily checked by measuring OP; that is, if OP measures 1000 units, the angle is correctly plotted.

Measuring Angles. These two methods may also be used in measuring angles. For example, to measure the $\angle QOP$ on page 232 by the tangent method, we lay off OQ as 1000 units, and construct the perpendicular QP. If $QP = 1080.5$ units, then

$$\tan O = \frac{1080.5}{1000} = 1.0805,$$

and from the table, $\angle O = 47° 13'$.

Similarly, to measure the $\angle POQ$, above, by the sine-cosine method, we lay off OP as 1000 units, and construct the perpendicular from P to l. If QP then measures 592.0 units and OQ measures 805.9 units, we have

$$\sin O = \frac{592.0}{1000} = 0.5920 \quad \text{and} \quad \cos O = \frac{805.9}{1000} = 0.8059.$$

From the tables of sines and cosines, $\angle O = 36° 18'$, the values as found from the two tables checking each other.

Exercises. Plotting and Measuring Angles

Plot each of the following angles by both the tangent method and the sine-cosine method:

1. 12° 15′.	3. 63° 38′.	5. 54° 36′.	7. 140° 40′.
2. 47° 20′.	4. 15° 11′.	6. 68° 22′.	8. 165° 20′.

In these exercises the student should use an accurate scale and a hard lead pencil sharpened to a fine point.

9. Construct a triangle with a base 2 in. long and with base angles of 47° 10′ and 106° 3′. Compute the lengths of the other two sides of the triangles by trigonometric methods, and check the results by measuring the sides.

10. Draw any convenient angle and measure it by the tangent method. Check your work by using the sine-cosine method.

11. Draw an oblique triangle with no two sides equal and measure all three angles by the tangent method. If the sum of the angles is not 180°, as it should be, find the source of the error by using the sine-cosine method.

12. Draw an irregular pentagon and measure the size of each angle by the tangent method. If the sum of the angles is not 540°, find the source of the error as in Ex. 11.

13. By plotting the five central angles in successive order about the center, construct a regular pentagon which shall have a radius of 3 in.

14. Using any convenient length of side, construct an equilateral hexagon in which each exterior angle in two opposite pairs shall contain 72°.

15. Inscribe a regular hexagon in a circle with a radius of 3 in. and measure each of the exterior angles by the sine-cosine method. Check your work by computing the exterior angles.

16. In measuring an angle that is close to 90° by the tangent method, what difficulties would be encountered? Would the sine-cosine method be better? Explain your answer.

17. After walking some distance in a straight line, a man is $\frac{1}{2}$ mi. to the north and $\frac{1}{4}$ mi. to the east of the point from which he started. Draw a figure to show the direction which the man took, and by using the tangent method find the angle which his path makes with the true north-and-south line.

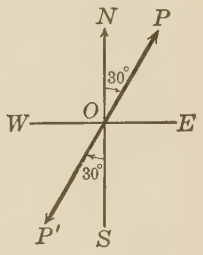
18. Starting at a point A, a man walks due east for a distance of 1240 ft. to B. If he then turns to the left through an angle of 56° 28′ and walks in a straight line for a distance of 1725 ft. to C, how far is he from his starting point? Draw the figure to scale and find by the sine-cosine method the angle which AC makes with AB. Through what angle must the man turn at C in order to reach A by the most direct route?

19. Two inaccessible points P and Q can both be seen from the points A and B, which are 2000 ft. apart. From A the lines of sight to P and Q make angles of 65° 10′ and 39° 53′ respectively with AB, while from B the angles to P and Q are 28° 14′ and 70° 23′ respectively. What is the distance from P to Q? Plot the figure and check the result by measuring PQ.

20. In making a survey from left to right around a four-sided field, a surveyor starting at A finds that the side AB is 1240 ft. long. At B he turns an angle left of 80° 20′, and finds that BC is 1060 ft. long. At C he turns an angle left of 76° 15′, and finds that CD is 1625 ft. long. From D the fourth side runs to the starting point A. Plot the survey, and from the data compute the size of each angle of the field, the length of the fourth side DA, and the angle which the surveyor would have to turn at D in running the line DA.

21. Using any convenient scale, plot the following meander survey along a stream: Starting point, A; $AB = 300$ ft., $\angle B = 15°\ 30'$, right; $BC = 475$ ft., $\angle C = 23°\ 45'$, left; $CD = 416$ ft., $\angle D = 44°\ 20'$, left; $DE = 600$ ft. Draw the "closing line" AE, and by measuring find the length of AE and the angle which it makes with AB. Check your work by drawing perpendiculars from B, C, and D to AE and computing AE.

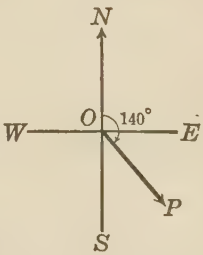
Bearing and Azimuth. In surveying we usually speak of the line between two points on the earth's surface as a *course.* The acute angle which a course makes with a meridian (that is, with a north-and-south line) is called the *bearing* of the course.



Thus, in this figure where NS represents the meridian, the bearing of the course $P'P$ is either $\angle NOP$ or $\angle SOP'$, according to the direction in which we are facing. That is, if we are facing P at O, the bearing is $\angle NOP$, which is designated as the bearing N. 30° E. (north 30° east). If we are facing P', the bearing is $\angle SOP'$, which is designated as the bearing S. 30° W. (south 30° west).

It should be observed that in giving the bearing of a course, the acute angle between it and its meridian is always used; that is, the bearing of a course is always less than 90°.

The direction of a course may also be designated by the size of the angle, read continuously from 0° to 360°, between the meridian and the course. Direction so measured is called *azimuth.*



Azimuth is read from either the north or the south as the reference point. Formerly, the more general custom was to use south as the zero point, but in certain cases it is better for practical reasons to use north. In our work we shall always take north as 0°; that is, the azimuth of the course OP in the last figure above is 140°, usually read "azimuth 140°."

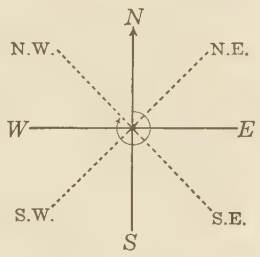
The bearing of the course OP in the second figure is S. 40° E.; that is, azimuth 140° is the same as the bearing S. 40° E.

Similarly, in the first figure above, the azimuth of the course OP is 30°, while that of the course OP' is 210°, or 180° greater than the other.

In mathematics we regard an angle generated by a clockwise rotation as negative. Since in reading azimuth we always use the clockwise rotation, we do not need to consider the signs.

Reducing Azimuth to Bearings. When the direction of a course is expressed as azimuth, it is frequently convenient to reduce it to a bearing, although in field work it is usually easier to read azimuths than bearings.

For an azimuth of a degrees it is evident from the accompanying figure that between 0° and 90° the corresponding bearing is a N. E. bearing of a degrees; between 90° and 180° the corresponding bearing is a S. E. bearing of $180 - a$ degrees; between 180° and 270° it is a S. W. bearing of $a - 180$ degrees; and between 270° and 360° it is a N. W. bearing of $360 - a$ degrees.



The reduction of a bearing to an azimuth is performed in a similar way. That is, a N. E. bearing of b degrees is an azimuth of b degrees; a S. E. bearing is an azimuth of $180 - b$ degrees; a S. W. bearing is an azimuth of $180 + b$ degrees; and a N. W. bearing is an azimuth of $360 - b$ degrees.

Exercises. Bearings and Azimuths

Reduce the following azimuths to bearings:

1. 47° 30′.	4. 40° 15′.	7. 84° 11′.
2. 202° 10′.	5. 272° 8′.	8. 265° 12′.
3. 165° 16′.	6. 187° 39′.	9. 342° 25′.

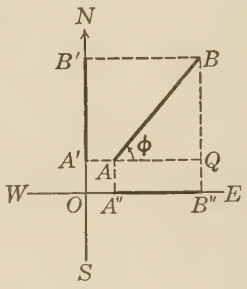
Reduce the following bearings to azimuths:

10. N. 27° 10′ W.	12. S. 12° 20′ W.	14. S. 45° 0′ E.
11. S. 68° 22′ E.	13. S. 79° 12′ W.	15. N. 88° 12′ W.

16. If from a course bearing N. 72° E. an angle of 27° is turned to the right, what is the bearing of the new course? What is the azimuth?

17. If the angle turned in Ex. 16 had been 96° left, what would have been the bearing? the azimuth?

Projection. In surveying we frequently need to use the projection of a course upon a north-and-south line or upon an east-and-west line. Thus, in this figure the line $A'B'$, the segment cut off by the perpendiculars to ON from A and B, is called the *projection* of the course AB upon ON. Similarly, $A''B''$ is the projection of AB upon OE.



If we produce $A'A$ to intersect BB'' at Q, it is evident that $\angle QAB$, or ϕ, is the angle which BA, if produced, would make with the line OE. Since $A''B'' = AQ$ and $A'B' = QB$, it is evident that we can easily find the lengths of the two projections $A''B''$ and $A'B'$ by using cos ϕ and sin ϕ respectively.

That is, in finding the length of $A''B''$ we have $A''B'' = AB \cos \phi$. Similarly, in finding the length of $A'B'$, we have $A'B' = AB \sin \phi$.

Exercises. Projection

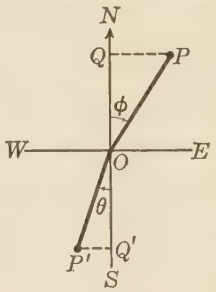
1. What is the length of the projection of a line 5.2 in. long upon a line with which it makes an angle of 27° 30′? an angle half as large? an angle twice as large?

2. In traveling 2.5 mi. upon a road which rises at a uniform slope of 8° 45′ with the horizontal, what is the horizontal distance traveled? What is the difference in elevation in feet between the two ends of this distance?

3. From a point A a railroad runs upward at a uniform slope of 4° 25′ to a point B, which is 2875 ft. from A, measured along the tracks. From B the road rises upward at a uniform slope of 6° 40′ to C, which is 3680 ft. from B, measured along the tracks. What is the horizontal distance from A to C? What is the difference in elevation between A and C?

4. If an automobile is traveling at 25 mi./hr. along a straight road that runs N. 43° 15′ W., how fast is the car moving north? How fast is the car moving west?

Latitude and Departure. In traveling from O to P on the course OP here shown, which has a N.E. bearing (or an azimuth less than 90°), it is evident that we are going north the distance OQ and east the distance QP.



Similarly, in traveling from O to P' on the course OP', which has a S.W. bearing (or an azimuth between 180° and 270°), it is evident that we are going south the distance OQ' and west the distance $Q'P'$.

The distances OQ and QP are called the *northing* and the *easting* respectively of the point P, and the distances OQ' and $Q'P'$ are called the *southing* and the *westing* respectively of the point P', each being referred to O as the starting point.

Instead of using the terms "northing" and "southing," we also speak of OQ and OQ' as the *latitude* of the point, north latitudes being considered positive and south latitudes negative. Similarly, instead of "easting" and "westing," we speak of QP and $Q'P'$ as the *departure* of the point, east departures being considered positive and west departures negative.

That is, the latitudes of points on N.E. and N.W. bearings are positive, while those of points on S.E. and S.W. bearings are negative. Similarly, the departures from N.E. and S.E. bearings are positive, while those from S.W. and N.W. bearings are negative.

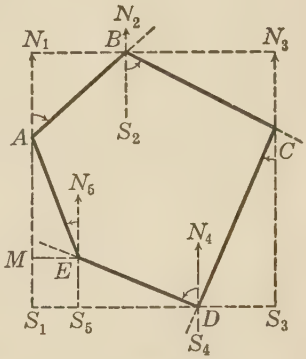
Since the latitude and departure of a point are the projections upon the meridian through the starting point and upon the east-and-west line through the terminal point respectively, it is evident from the discussion on page 238 that

1. *The latitude of a point is the product of the length of the course and the cosine of the bearing;*

2. *The departure of a point is the product of the length of the course and the sine of the bearing.*

That is, the latitude is the side adjacent to the bearing, and hence we use the cosine of the bearing. Similarly, the departure is the side opposite the bearing, and hence we use the sine.

Closed Traverse. In surveying a piece of land like the field $ABCDE$ here shown the survey will pass completely around the field, and we have what is known as a *closed traverse.* Although any corner of the field may be selected as the starting point, it is usually more convenient to record the data in such a way that either the most western or the most eastern station comes first.



In the figure the lines N_1S_1, N_2S_2, $\cdots$ indicate the meridians through the successive vertices at which the transit is set up. The bearings of the successive courses AB, BC, $\cdots$ are $\measuredangle N_1AB$, S_2BC, $\cdots$, as indicated by the curved arrows. The latitude of the course AB is AN_1, which is positive, and the departure of the course AB is N_1B, which is also positive. Similarly, the latitude of the course BC is N_3C, which is negative, and the departure is BN_3, which is positive.

In starting from and returning to the point A it is apparent that any distance that is moved north is offset by a corresponding distance moved south; that is, the sum of the northings is equal to the sum of the southings. Similarly, the sum of the eastings is equal to the sum of the westings. In other words, the algebraic sum of the latitudes is zero, and so is the algebraic sum of the departures. This principle furnishes a convenient check upon the accuracy of the field work, and is useful in plotting the traverse.

The area inclosed by the traverse in the above case is evidently the area of the rectangle $N_1N_3S_3S_1$ minus the sum of the five right triangles N_1BA, BN_3C, $\cdots$, AEM and the rectangle MES_5S_1. These areas are readily found by first computing the latitudes and departures of the sides.

It is sometimes more convenient to measure only one bearing and then find the exterior angles formed by producing the sides successively. The bearing of any side can then be readily found from its exterior angle and the bearing of the preceding side; for example, the bearing of BC is evidently 180° minus the sum of the bearing of AB and the exterior angle at B.

Exercises. Latitudes and Departures

1. What is the latitude and departure of a course which is 1245 ft. long and has the bearing S. 38° 24′ W.? of one which is twice as long and has the bearing N. 76° 48′ E.?

Compute the latitudes and departures of the following courses:

2. 875.2 ft., N. 16° 25′ W.

3. 1276 ft., S. 87° 8′ E.

4. 1456 ft., N. 63° 20′ W.

5. 1232 ft., N. 0° 00′.

6. 896.5 ft., S. 45° 00′ E.

7. 18.24 mi., N. 24° 22′ E.

8. A man traveled S. 65° 30′ E. for a distance of 5 mi. He then turned 20° 10′ right, and traveled 5 mi. further. How far south and how far east was he from his starting point?

9. In Ex. 8, how far was the man in a direct line from the starting point, and in what direction?

10. For each of the points in the following traverse compute the total latitude (the algebraic sum of the latitudes of the separate points) and the total departure: *AB*, 654.2 ft., N. 27° 30′ E.; *BC*, 587.6 ft., S. 78° 15′ E.; *CD*, 456.4 ft., N. 8° 45′ E. Plot the traverse to a convenient scale.

11. In the following notes of a preliminary highway survey the first column shows the numbers of the transit points, the second the lengths of the courses, and the third the bearings:

Point No.	Length of Course	Bearing
1		
	1200 ft.	N. 20° W.
2		
	900 ft.	N. 11° E.
3		
	1750 ft.	N. 66° W.
4		
	825 ft.	N. 0°
5		

Copy the table, adding four columns at the right to show (1) the latitude, (2) the total latitude, (3) the departure, and (4) the total departure for each of the transit points. Plot the traverse to a convenient scale.

12. From the following notes recorded in running the meander line of a stream prepare a table, adding five columns showing respectively the bearings, the latitudes, the total latitudes, the departures, and the total departures: *AB*, 400 ft., azimuth 15° 15′; *BC*, 2400 ft., azimuth 342° 15′; *CD*, 1500 ft., azimuth 35° 3′; *DE*, 300 ft., azimuth 87° 12′; *EF*, 1000 ft., azimuth 320° 20′. Plot the traverse and compute the length and bearing of the closing line *AF*.

13. Arrange the following notes of a closed traverse in tabular form (including four columns for latitudes and departures): *AB*, 571.6 ft., N. 20° E.; *BC*, 330.0 ft., S. 70° E.; *CD*, 660.0 ft., S. 10° E.; *DA*, 660.0 ft., N. 70° W. Plot the traverse and compute the area to the nearest 0.01 A. (1 A. = 43,560 sq. ft.)

14. The following field notes were made in surveying the campus at University Farm, St. Paul, Minnesota. The table shows the length of each course, the angle turned (right R. and left L.) at the end of each course, and the bearing of the first course. Copy the notes, filling in the columns showing the bearings, and add four columns at the right of the table to show (1) the latitude of each point, (2) the total latitude, (3) the departure of each point, and (4) the total departure:

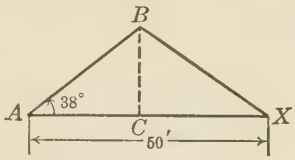
Course	Length	Angle	Bearing
AB	633.72 ft.		N. 0° 00′
BC	321.14	0° 00′	
CD	523.70	91° 03′ R.	
DE	105.28	119° 52½′ R.	
EF	246.29	31° 14¾′ L.	
FG	135.75	0° 04′ L.	
GH	189.42	2° 26½′ R.	
HI	213.55	29° 59′ R.	
IJ	366.85	41° 42½′ R.	
J = *A*		106° 16′ R.	

Plot the traverse to a convenient scale and find the area inclosed to the nearest 0.01 A.

Exercises. Test on Chapter XI

1. With a transit placed 9 ft. above the level of the water, a surveyor on one bank of a river 2650 ft. wide finds that the angle of elevation of the top of a cliff which rises vertically on the opposite bank is 5°. What is the height of the cliff?

2. In making the plans for a roof which is to have a span (AX in the figure) of 50 ft. and an angle of slope of 38°, what length (AB) should be allowed for the rafters? How high is the peak of the roof above the eaves?



3. A straight road runs at an average gradient of 3° for a distance, measured along the road, of $2\frac{1}{2}$ mi. How many feet higher is one end of the road than the other?

4. After 1600 ft. of cable has been unreeled in sending up a captive balloon, how high is the balloon when the cable makes an angle of 82° with the ground?

The cable will, of course, sag considerably; if we assume it to be straight, however, the result will be sufficiently accurate for our purposes.

5. Find to the nearest 0.01 in. the length of the side of the largest regular decagon that can be inscribed in a circle that is 6.84 in. in diameter.

6. How long is the projection of a 10-inch line upon another line with which it makes an angle of 23° 15′?

7. Express azimuth 235° 45′ as a bearing; express the bearing N. 25° 14′ W. as azimuth.

8. Show by a figure why latitude is called "northing" or "southing"; why departure is called "easting" or "westing."

9. In testing the accuracy of the field notes of a closed traverse, what check is the most convenient?

10. Draw a figure to show how the functions of an obtuse angle may be found from those of an acute angle.

Exercises. General Review

Letting $x = 3$, $y = 2$, $a = 4$, *and* $b = 5$, *find the value of*

1. $3x^2 - 2xy$. **3.** $xy^3 - 2x^3y$. **5.** $(a + b + x)^2$.

2. $2a^2b + ab^2$. **4.** $a^2x + 4b^2y$. **6.** $a^2 + b^2 + x^2$.

Letting $A = x^3 - 2x^2 - 29x - 42$ *and* $B = x^2 - 4x - 21$, *find*

7. $A + B$. **8.** $A - B$. **9.** AB. **10.** $A \div B$.

11. In checking the solution of a triangle, why is it desirable to use a formula that was not used in the solution?

12. Find the value of $400 \times 1.04^6 \div 1.04^3$.

13. Solve for n the formula $C = \dfrac{en}{R + nr}$, which is used in the study of electricity.

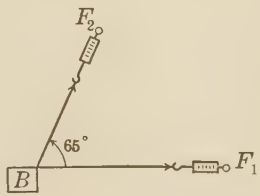
14. In Ex. 13 find n when $C = 0.38$, $R = 6$, $r = 3$, and $e = 1.04$.

15. Solve the equation $2(x^2 - x) = 2x^2 - 16$.

16. What is the fiftieth term of the series $a + b$, $2a + 3b$, $3a + 5b$, $\cdots$? What is the sum of ten terms?

17. Find the sum of six terms and also of n terms of the series $(a - b)$, $(a - b)^2$, $(a - b)^3$, $\cdots$.

18. When two spring balances are attached to a heavy block B, as here shown, the block starts to move when the balances F_1 and F_2 indicate 3.8 lb. and 2.6 lb. respectively. What single force must be applied to the block to produce the same effect as the two forces, and in what direction from the first force (F_1) must it be applied?



19. If x varies directly as y, and if $y = 2$ when $x = 24$, what is the value of x when $y = 15$?

20. Solve Ex. 19 when x varies inversely as y.

21. When a flagpole 58 ft. high casts a shadow 39.2 ft. long, what is the angle of elevation of the sun?

CHAPTER XII

PRACTICAL MENSURATION

Purpose of the Work. The ability to compute the areas of plane figures and the lateral areas and volumes of solid figures is important in industry and the sciences. We have already made considerable use of the formulas for finding the areas of the more common plane figures and the volumes of the simpler solids. Before taking up the more complicated cases, we shall first review the more familiar ones.

Symbols used in the Formulas. In writing formulas it is important that a systematic scheme of notation be used, and it is also desirable that, as far as possible, the symbols employed shall suggest the words for which they stand.

Although we shall consider each formula in connection with the figure to which it refers, the following list, which shows what the letters represent, will be found convenient for reference:

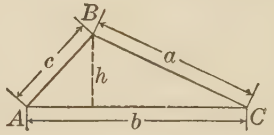
a, b, c = sides of $\triangle ABC$
a, b = radii of ellipse
A = area
b, b' = lengths of bases
B = area of base
C = circumference
d = diameter; diagonal
e = element; lateral edge
E = area of ellipse
F, F' = foci of ellipse
h = height (altitude)
k = length of chord
l = length; slant height
L = lateral area
M = area of midsection
p = apothem; perimeter
Q = area of quadrilateral
r = radius
s = side; also $s = \frac{1}{2}(a + b + c)$
S = area of curve surface
T = area of triangle
$\pi = 3\frac{1}{7}$ or 3.1416
$\frac{1}{\pi} = \frac{7}{22}$ or 0.3183

It will be seen that some of the letters in this list are used for several purposes; for example, A, B, and C may represent the vertices of a triangle in one case, or A may be used for area, B for the area of the base of a solid figure, and C for the circumference of a circle.

Area of a Triangle. Of the four formulas which are commonly used in computing the area of a triangle, the simplest is the familiar one

$$T = \tfrac{1}{2}\, bh,$$

which is used in finding the area T when the base b and the height h are the given parts of the triangle.



When the three sides are known, we generally use the formula

$$T = \sqrt{s(s-a)(s-b)(s-c)},$$

where s represents the semiperimeter; that is, $s = \frac{1}{2}(a + b + c)$.

For example, in a case in which $a = 3$ in., $b = 4$ in., and $c = 5$ in., we have $s = \frac{1}{2}(3 + 4 + 5) = 6$, and $T = \sqrt{6 \cdot 3 \cdot 2 \cdot 1} = 6$ (sq. in.).

When two sides and the included angle are known, the most convenient formula is the trigonometric one

$$T = \tfrac{1}{2}\, bc \sin A.$$

For example, in a case in which $b = 9$ in., $c = 10$ in., and $\angle A = 34°$, we have $T = \frac{1}{2} \cdot 9 \cdot 10 \cdot \sin 34° = 45 \cdot 0.5592 = 25.1640$. Assuming an accuracy of measurement to the nearest 0.1 in., we give the area as 25.2 sq. in.

From the preceding formula we may derive another for use when two angles and the included side are known. For example, if we know $\measuredangle A$ and C and the side b, we have, as above,

$$T = \tfrac{1}{2}\, bc \sin A.$$

From pages 222 and 221 respectively, we know that

$$c = \frac{b \sin C}{\sin B},$$

and $$\sin B = \sin [180° - (A + C)] = \sin (A + C).$$

Hence $$T = \frac{b^2 \sin A \sin C}{2 \sin (A + C)}.$$

For example, if $\angle A = 33°$, $\angle C = 42°$, and $b = 10$ in., we have

$$T = \frac{10^2 \sin 33° \sin 42°}{2 \sin 75°},$$

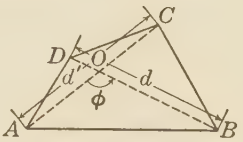
from which we can readily find the value of T by using logarithms.

Area of a Quadrilateral. One way of finding the area of a quadrilateral, such as $ABCD$ in the figure below, is to draw either diagonal, thus dividing the figure into triangles whose areas can be found by one of the formulas on page 246.

We may also develop a formula for the area Q in terms of the two diagonals (d and d' in the figure) and the angle (ϕ in the figure) included between them.

Using the formula $T = \frac{1}{2} bc \sin A$ (page 246) and remembering that $\sin (180° - \phi) = \sin \phi$, we may proceed as follows:

In $\triangle ABO$, $T = \frac{1}{2} AO \cdot BO \cdot \sin \phi$;
In $\triangle BCO$, $T = \frac{1}{2} BO \cdot CO \cdot \sin \phi$;
In $\triangle CDO$, $T = \frac{1}{2} CO \cdot DO \cdot \sin \phi$;
In $\triangle DAO$, $T = \frac{1}{2} DO \cdot AO \cdot \sin \phi$.



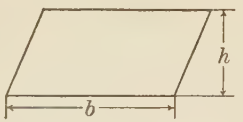
We therefore have

$$Q = \tfrac{1}{2}(AO \cdot BO + BO \cdot CO + CO \cdot DO + DO \cdot AO) \sin \phi$$
$$= \tfrac{1}{2}(AO + CO)(BO + DO) \sin \phi$$
$$= \tfrac{1}{2} AC \cdot BD \cdot \sin \phi.$$

That is, $$Q = \tfrac{1}{2} dd' \sin \phi.$$

Area of a Parallelogram. It is proved in geometry that the area of a parallelogram (including the rectangle, of course, as a special case) may be found from the formula

$$A = bh,$$

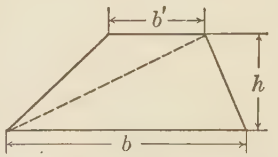


in which A represents the area, b the base, and h the height.

Area of a Trapezoid. The area of a trapezoid may be found by drawing either diagonal, thus dividing it into two triangles, as here shown. We then have

$$A = \tfrac{1}{2} bh + \tfrac{1}{2} b'h,$$

or $$A = \tfrac{1}{2} h(b + b'),$$



in which A represents the area, h the height, and b and b' the lower and the upper bases respectively.

Areas of Regular Polygons. We occasionally have to find the area of a regular polygon; that is, of a plane rectilinear figure which is both equilateral and equiangular.

It is proved in geometry that a circle can always be inscribed in or circumscribed about a regular polygon. The radius of the inscribed circle is called the *apothem* of the polygon, and the radius of the circumscribed circle is called the *radius of the polygon*. In the following formulas we shall use A for area, p for apothem, r for radius, and s for side.

The regular polygons most often found in science and practical construction are the *pentagon* (with five sides), the *hexagon* (with six sides), and the *octagon* (with eight sides). The formulas most often used in connection with them are as follows:

Pentagon, $A = 1.7205\ s^2$, $r = 0.8506\ s$, $p = 0.6882\ s$;

Hexagon, $A = 2.5980\ s^2$, $r = s$, $p = 0.8660\ s$;

Octagon, $A = 4.8284\ s^2$, $r = 1.3065\ s$, $p = 1.2071\ s$.

Exercises. Areas of Rectilinear Figures

1. Construct a triangle in which the sides are respectively 2.5 in., 3.7 in., and 4.3 in. long. Measure any altitude and find the area of the triangle. Check your work by means of the second formula on page 246.

In many kinds of practical work the more convenient decimal subdivisions of the foot and inch are now extensively used. If the students do not have rulers graduated to tenths of an inch, a temporary substitute can be devised from graph paper ruled with ten squares to the inch.

2. Find a formula for the area T of an equilateral triangle in terms of the side s; in terms of the height h.

3. Construct an equilateral triangle that shall have an area of 3 sq. in. Check your work by measuring the base and the altitude and computing the area from these measurements.

4. If a square and a rectangle have each a perimeter of 576 in., and if the length of the rectangle is five times its width, which has the greater area and how much greater is it?

5. Find the area of a parallelogram in which two adjacent sides are respectively 3 in. and 4 in. long and the included angle is 35° 28′. Check your answer by drawing the figure, measuring the altitude, and computing the area.

The area is first found, of course, by trigonometric methods.

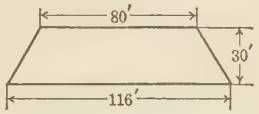
6. If the diagonal of a rectangle is 3.7 in. long and makes an angle of 31.5° with the base, what is the area of the rectangle? Check your answer by drawing the figure, measuring the sides, and computing the area.

7. In a rhombus with diagonals of 2.5 in. and 1.7 in. find the length of each side, the size of each angle, and the area.

A *rhombus* is a parallelogram in which all the sides are equal.

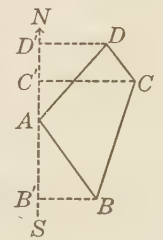
8. Two rectangular city lots have the same depth, the frontage of the first is twice that of the second, and their combined frontage is equal to their common depth. Find the ratio of the areas of the two lots and find also the ratio of the perimeters.

9. Find the area of the outer face of a concrete retaining wall which is 30 ft. high, 80 ft. wide at the top, and 116 ft. wide at the bottom.



10. What is the cross-section area of the largest square timber that can be sawed from a round log with a diameter of 2 ft. 3 in.? from a log that has only half this diameter?

11. In surveying the four-sided field $ABCD$ here shown, a surveyor first laid off the north-and-south line NS through A. He then found by measuring that $BB' = 6$ rd., $CC' = 10$ rd., $DD' = 7$ rd., $B'A = 8$ rd., $B'C' = 12$ rd., and $C'D' = 4$ rd. Find the area of the field in acres (1 A. = 160 sq. rd.).



12. Describe another method of finding the area in Ex. 11, drawing a figure to illustrate it.

13. Find the area of a regular hexagon inscribed in a circle that has a diameter of 2.8 in. Draw the figure.

Area of a Circle. The area of a circle can be readily found by several formulas. For example, in terms of the circumference C and the radius r, we have the formula

$$A = \tfrac{1}{2} Cr.$$

Since $C = 2\pi r = \pi d$, we can easily develop more convenient formulas for the area by observing that

$$A = \tfrac{1}{2} Cr = \tfrac{1}{2} \cdot 2\pi r \cdot r,$$

and

$$A = \tfrac{1}{2} Cr = \tfrac{1}{2} \cdot \pi d \cdot \tfrac{1}{2} d;$$

whence

$$A = \pi r^2 \quad \text{or} \quad A = \tfrac{1}{4} \pi d^2.$$

The value of π to four decimal places is 3.1416, but for our purposes we shall use $\frac{22}{7}$ as the value of π unless it is otherwise directed.

Area of a Circular Ring. In machine-shop work it is often necessary to find the area of a circular ring like the one here shown. Obviously, this area is the difference between the areas of the two circles; that is,

$$A = \pi r^2 - \pi r'^2;$$

whence

$$A = \pi(r + r')(r - r').$$

Area of a Sector. The area of a sector of a circle can be found if we know the radius of the circle and the size of the central angle, or if we know the radius and the length of the chord of the sector.

If we let θ represent the number of degrees in the central angle, it is evident that

$$A = \frac{\theta}{360} \pi r^2.$$

Since the bisector of $\angle O$ is the perpendicular bisector of the chord k of the sector, we also have

$$\sin \tfrac{1}{2} \theta = \frac{\frac{1}{2} k}{r},$$

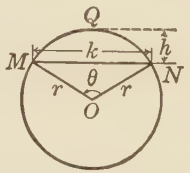
or

$$\sin \tfrac{1}{2} \theta = \frac{k}{2r}.$$

We can thus find θ and then use the above formula for A.

Area of a Segment. As shown in this figure, the area of the segment MNQ of the circle can be found by subtracting the area of the $\triangle MON$ from that of the sector $MONQ$. That is, we have

$$A = \frac{\theta}{360}\pi r^2 - \tfrac{1}{2} r^2 \sin\theta.$$



A more convenient formula in terms of the chord k and the height h of the segment is

$$A = \tfrac{2}{3} hk + \frac{h^3}{2k}.$$

Although this formula for the segment gives only approximate results, they are sufficiently accurate for our purposes.

Exercises. Measuring the Circle

1. A sign painter who is laying out a clock face wishes to show the time as 20 min. past 7 o'clock. Draw a circle with a radius of 2 in. and show the position of the hands.

2. In a circle 12.5 in. in diameter find the length of the chord of a sector having a central angle of 32.6°.

3. When a sector of 28.2° (that is, one in which the central angle contains 28.2°) is cut from a circle 6.8 in. in diameter, by how many square inches is the area of the circle reduced?

4. A water main is to be laid to two branch pipes which have diameters of 10 in. and 16 in. respectively, and the diameter of the main must be such that the area of its cross section equals the combined cross-section areas of the branch pipes. Find the diameter of the main to the nearest $\frac{1}{4}$ in.

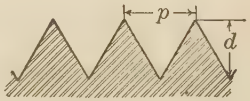
5. When the water in a horizontal circular pipe with a diameter of 20 in. is 7.5 in. deep, what is the approximate cross-section area of the "wetted segment"?

6. If the circumference of one circle is twice that of another, how does the area of an equilateral triangle constructed on the diameter of the first as a side compare with that of an equilateral triangle constructed on the diameter of the second?

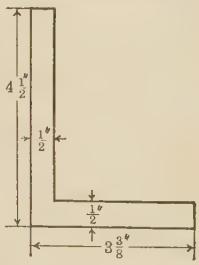
Exercises. Plane Figures

1. If the drive wheels of a locomotive are 6 ft. in diameter and make 1722 revolutions while the locomotive is traveling a distance of 6 mi., what is the distance lost through the slipping of the wheels on the rails?

2. This figure shows that the cross section of a V-thread on a screw is an equilateral triangle. The distance p between successive threads is called the "pitch" of the thread and the distance d its "depth." Find the depth of a thread which has a pitch of 0.125 in.; of 0.25 in.



3. Find the number of square inches in the cross section of the L-beam here shown.



4. A machine for planing iron plates planes a surface 1 in. wide and 9 ft. long in 1 min. At the same rate per square inch, how long will it take to plane a plate 12 ft. long and 6 in. wide, allowing 28 min. for adjusting the machine?

5. From one end of a rectangular sheet of tin 12 in. long a tinsmith cut off a square piece. If the area of the remaining part was 36 sq. in., what was the width of the sheet?

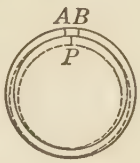
6. If squares are inscribed in two circles with radii of 6 in. and 18 in. respectively, what is the ratio of the areas of the squares? of the perimeters?

7. The circumferences of two cylindric steel shafts are 7 in. and 3.5 in. respectively. The area of the cross section of the first is how many times that of the second?

8. At 120 lb./sq. in., what is the total pressure on the circular head of a piston 5.6 in. in diameter?

9. If the diameter of the hole in an iron washer is 2.75 in., and the width of the metal ring is 0.75 in., what is the area of one face of the washer?

10. In turning a piston ring for an engine it is customary to make a larger ring than is needed in the cylinder, the outside diameter of the ring being usually made 1.5% longer than the diameter of the cylinder. A piece like AB is then cut out, and the ring is drawn together at P, as shown by the dotted lines, and fitted into place. What diameter should be used for a ring for a cylinder 4 in. in diameter, and what length (AB) should be cut off to make the ring fit?



11. A hollow cylindric supporting column has an outside diameter of 7 in. and an inside diameter of 5 in. What is the metallic cross-section area of the column?

12. Two points on a railroad curve which is an arc of a circle are 400 ft. apart in a straight line. If the midpoint of the arc between the two points is 28 ft. from the midpoint of the chord which joins the points, with what radius was the curve originally laid out?

13. A pipe with an inside diameter of 8 in. can carry how many times as much water as a 1-inch pipe? as a 2-inch pipe? It can carry what part as much as a 16-inch pipe?

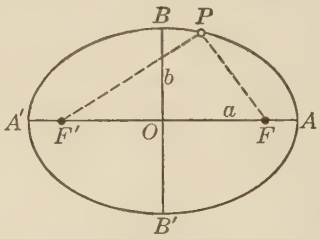
14. If a farmer wishes to inclose as large a grazing area as possible with 100 rd. of wire fencing, should he build the fence in the form of a square or in the form of a circle? What is the difference between the areas inclosed?

15. A draftsman wishes to draw a regular pentagon that will be 1.8 in. on a side. What radius should he use in drawing the circle in which to inscribe the pentagon? What will be the area of the pentagon?

16. If the top part of a kite is a semicircle with a diameter of 20 in., and the bottom part is a triangle with a height of 28 in., what is the area of the kite?

17. The area of a sector which has a central angle of 18° is what part of the area of the circle?

Ellipse. In our study of graphs on page 62 we learned that the *ellipse*, the curve which has the shape shown in the figure at the right, is the graph of a certain type of quadratic equation. The ellipse can also be derived by cutting a right circular cone by a plane which is oblique to the axis, and it is therefore known as one of the *conic sections.*



The other three curves illustrated on page 62, the circle, the parabola, and the hyperbola, can also be derived by letting a plane cut a right circular cone at different angles. For the present, however, the only conic section that we need to consider is the ellipse.

The line segment $A'A$ in the above figure is called the *major axis* of the ellipse, and the segment $B'B$ is called the *minor axis.* The letters a and b are customarily used to designate half of the major and minor axes respectively. The points F and F' are called the *foci* of the ellipse.

In analytic geometry it is shown that an important property of the ellipse is that the sum of the distances from the foci to any point on the curve is always constant, and that this constant sum is equal to twice the major semiaxis; that is, $FP + F'P = 2\,a$.

The area E of an ellipse may be found from the formula

$$E = \pi ab.$$

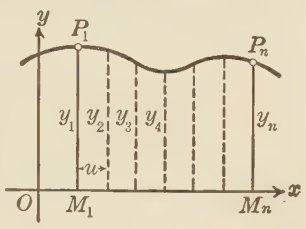
A circle may be regarded as a special case of the ellipse; that is, as one in which the axes are equal. As an ellipse approaches a circle, it is evident that b becomes more nearly equal to a, until, when $b = a$, we have the formula for the area of a circle, $A = \pi r^2$.

A simple formula that will give a fairly close approximation to the perimeter p of the ellipse is

$$p = \pi \sqrt{2(a^2 + b^2) - \frac{(a - b)^2}{2.2}}.$$

The development of these two formulas for the ellipse requires mathematics beyond the scope of this book, so that for our present purposes we may simply assume that they are capable of proof.

Simpson's Rule for Areas. It is occasionally necessary in practical work to find the approximate area of a figure which is bounded on one side by an irregular curve like the one shown in the accompanying figure.



By drawing and measuring an odd number of ordinates (y_1, y_2, y_3, $\cdots$, y_n) having a common distance u between them, as here shown, we may compute the approximate area under the curve between the ordinates y_1 and y_n by means of the formula

$$A = \tfrac{1}{3}u[(y_1 + y_n) + 2(y_3 + y_5 + y_7 + \cdots) + 4(y_2 + y_4 + y_6 + \cdots)].$$

This formula expresses what is known as *Simpson's Rule*, so named from its inventor. Since there must be an odd number of ordinates, n must be an odd integer. The odd-numbered ordinates are therefore y_1, y_3, y_5, $\cdots$, y_n, while the even-numbered ordinates are y_2, y_4, y_6, $\cdots$, y_{n-1}.

Exercises. Simpson's Rule

1. If in a curve like the one above the ordinates y_1 to y_7 are respectively 2.00 in., 1.92 in., 1.76 in., 1.62 in., 1.78 in., 1.90 in., and 1.85 in., and if $u = 0.40$ in., what is the approximate area under the curve between y_1 and y_7?

2. Suppose that the above figure represents a field bounded on one side by a stream, and that at intervals of 20.0 ft. the ordinates are found to be 98.2 ft., 106.7 ft., 128.4 ft., 126.2 ft., 124.6 ft., 122.7 ft., 108.4 ft., 108.4 ft., 106.3 ft., 108.5 ft., and 110.6 ft. What is the area of the field?

3. On squared paper draw an undulating (wave-like) curve, and by measuring the ordinates find the approximate area under the curve between the ordinates at $x = 4$ and $x = 18$. Check your result by counting the squares.

In counting the squares cut by the curve, exclude any square that does not lie at least half within the figure, count as a half square one that is about evenly divided, and count as a full square one that lies more than half inside.

Exercises. Areas of Plane Figures

1. Find the area in acres (1 A. = 43,560 sq. ft.) of a triangular field whose sides are 1000 ft., 1200 ft., and 1320 ft.

2. A straight road cuts across the corner of a farm for a distance of 1470 ft., making angles of 33° and 55° with the adjacent sides of the farm. What is the area in acres of the triangular piece of land thus cut off?

3. What is the area in square feet of the cross section of a trapezoidal flume 32 in. wide at the top, 28 in. wide at the bottom, and 10 in. deep?

4. How many acres are there in a four-sided piece of ground in which the diagonals are 60 rd. and 66 rd. in length and the angle between the diagonals is 87°?

5. Find the inside cross-section area of a drain pipe with an inside diameter of 5 in.; of 6 in.; of 8 in.; of 10 in.

6. What is the area of a segment with a height of 1 in. cut from a circle whose diameter is 8 in.?

7. In fitting a 12-inch circular furnace pipe into an elliptical hole the pipe was flattened to an ellipse whose minor axis was 9 in. In what way and by how much was the cross-section area affected at that point?

8. A farmer wishes to find the approximate area of a four-sided field in which one corner is a right angle. He finds that the sides adjacent to the right angle measure 44 rd. and 50 rd. respectively, and that the diagonal from the vertex of the right angle makes an angle of 45° with the shorter side and is 70 rd. long. What is the area of the field to the nearest 0.1 A.?

9. If a rectangular field containing 20 A. is 32 rd. wide, what are the dimensions of a field of exactly the same shape and containing 48 A.?

Since the fields are similar, their areas are proportional to the squares of any two corresponding sides.

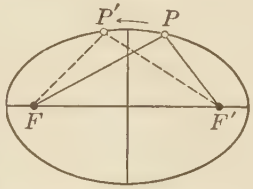
10. What is the area of the metallic cross section of a 3-inch pipe in which the outside diameter is 3.375 in.?

11. If a race track is in the form of an ellipse in which the major axis is 1000 ft. and the minor axis is 400 ft. (each measured to the inside of the track) what is the area inclosed?

The answer should be given correct to the nearest square foot.

12. In Ex. 11, if the track is uniformly 30 ft. wide, what are the inside and outside perimeters of the track?

13. A simple practical method of drawing an ellipse is as follows: Fasten the ends of a piece of string to two pins placed at the foci F and F', draw the string taut by means of a pencil point P, and then move the pencil about F and F', as shown in this figure. Draw an ellipse by this method, measure the axes, and then find the area by the formula given on page 254.



14. In Ex. 13 find the area of the ellipse by Simpson's Rule (page 255). State the difference, if any, between the results.

15. Draw a figure similar to this but about four times as large. Draw any additional lines that are necessary in applying Simpson's Rule, and find the area by this method.

16. Two lengths of 6-inch circular stove pipe are to be joined together at right angles. Find the area of the elliptical section determined by the seams of the elbow. This is how much larger than the cross-section area of the pipe?

By using Simpson's Rule find the area between the curve and the x axis in each of the following cases:

17. Ordinates: 14 ft., 16 ft., 17 ft., 15 ft., 13 ft., 12 ft., 14 ft.; common distance between successive ordinates, 4 ft.

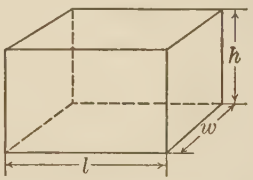
18. Ordinates: 0 in., 1.3 in., 2.4 in., 3.5 in., 4.6 in., 5.7 in., 6.8 in., 5.07 in., 4.06 in.; common distance, 2.4 in.

Solid Figures. In practical work it is often necessary to measure the capacities (the volumes) or the faces of various solid figures. For example, we may wish to find the capacity of a cylindric silo, the weight of a load of hay, or the surface of a hemispherical dome.

Many of the solids that we shall discuss are already familiar to the student. It will not, in general, be necessary to give definitions of the various geometric terms that are used because their meanings are either already familiar or will be apparent from the figures.

Volume of a Rectangular Solid. The solid that it is most often necessary to measure is the rectangular solid (also called a rectangular parallelepiped), such as is here shown. The formula for the volume V in terms of the length l, the width w, and the height h is

$$V = lwh.$$



In the case of the cube, a rectangular solid with equal edges e, we have

$$V = e^3.$$

Since the volume of a rectangular solid is the product of the number of square units in the base and the number of linear units in the height, the formula for V in terms of the area B of the base and the height h is

$$V = Bh.$$

This formula applies to all parallelepipeds, whether or not they are rectangular solids, but in our work we shall consider only the latter.

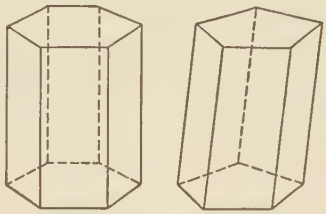
Exercises. Rectangular Solids

1. Taking $7\frac{1}{2}$ gal./cu. ft., write a formula for the capacity in gallons of a rectangular water tank that is l feet long, w feet wide, and d feet deep.

2. If the length, width, and height of a room are a, b, and c respectively, what is the total area of the walls and ceiling?

3. The outside dimensions of a closed wooden box are 8 in., 19 in., and 12 in., and the area of the total inside surface is 802 sq. in. Find the thickness of the wood used.

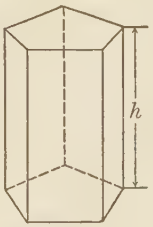
Prisms. In making excavations or in calculating the capacities of containers it is frequently necessary to measure a *prism*, — the solid which has two opposite faces that are congruent polygons in parallel planes and whose lateral (side) faces are parallelograms equal in number to the sides of a base.



The first of the accompanying figures shows a *right prism* which has hexagons for its bases. In such a prism the lateral faces are all rectangles. The second figure shows an *oblique prism* which has pentagons for its bases. In such a prism the lateral faces are oblique parallelograms.

Although both types of prisms are here mentioned for the sake of completeness, in practical work it is generally necessary to measure only right prisms; that is, those in which the lateral edges are perpendicular to the upper and lower bases.

Lateral Area of a Prism. It is evident that the lateral area of a prism is the sum of the areas of the lateral faces. Since in a right prism the rectangles all have the same height (the height of the prism) and the sum of their bases is the perimeter p of the prism, we may find the lateral area L of the prism by means of the formula



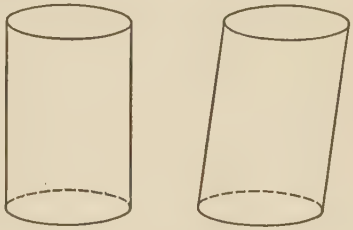
$$L = hp.$$

In the case of the oblique prism we may find the lateral area by multiplying the perimeter of a section perpendicular to a lateral edge of the prism by the length of a lateral edge.

Volume of a Prism. As in the case of the rectangular solid, it is evident that the volume of a prism is the product of the number of square units (B) in the area of the base and the number of linear units (h) in the height. The volume of any prism, whether right or oblique, is therefore given by the formula

$$V = Bh.$$

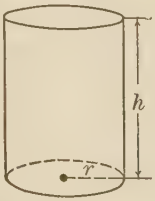
Cylinders. Corresponding to the right prism and the oblique prism, we have, as shown in these figures, the *right cylinder* and the *oblique cylinder* respectively. In both cases the lateral surfaces of these solids are curve surfaces, and the bases lie in parallel planes.



Although the bases of a cylinder may be a curvilinear figure of any shape, in our work we shall need to consider only cylinders in which the bases are circles, — the so-called *circular cylinders.*

The projection of a circle upon a plane oblique to the plane of the circle is an ellipse. The two figures shown above represent circular cylinders, but since they are seen obliquely, the bases in both cases are represented by ellipses instead of circles.

Area of the Curve Surface of a Cylinder. If we consider a right circular cylinder in relation to a right prism, we see that the perimeter p of the base of the right prism corresponds to the circumference $2\pi r$ of the base of the cylinder. The formula for the area S of the curve surface of the cylinder may therefore be written as



$$S = 2\,\pi r h.$$

In the case of the oblique cylinder we may find the area of the curve surface by multiplying the perimeter of a section perpendicular to an *element* (the line corresponding to the lateral edge of a right prism) by the length of the element.

Volume of a Cylinder. In finding the volume of a circular cylinder it is evident that B, the area of the base, in the prism formula $V = Bh$ may be replaced by πr^2; that is, we have

$$V = \pi r^2 h.$$

The general formula $V = Bh$ may be used in finding the volume of any cylinder regardless of the shape of the base. In the case of circular cylinders, however, the above formula is usually more convenient.

Exercises. Prisms and Cylinders

1. Find the area of the total surface of a right prism which is twice as long as it is thick and which has a square base with an area of 5.8 sq. in.

2. A glass jar with an inside diameter of 5 in. is partly filled with water. If a stone placed in the jar causes the water level to rise $1\frac{1}{2}$ in., what is the volume of the stone?

This is a convenient way of finding the volume of an irregular solid.

3. On the main floor of a bank building are nine concrete supporting columns, each 18 ft. high with a regular octagonal cross section 15 in. on a side. Find the amount of surface to be covered in painting the columns.

4. A watering trough with a flat bottom and sloping sides is 32 in. long, 24 in. deep, 20 in. wide at the top, and 16 in. wide at the bottom. Allowing 7.5 gal./cu. ft., find the number of gallons of water that the trough will hold.

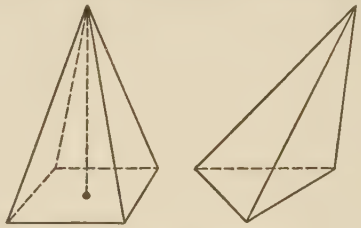
5. Find the difference between the area of the total surface of a right prism 8.5 in. high whose cross section is a regular hexagon inscribed in a circle 4.0 in. in diameter and a prism of the same height whose cross section is a regular octagon inscribed in a circle of the same size.

6. A tinsmith wishes to make 4 doz. cylindric cups, open at the top. The diameter of the base of each cup is to be 3 in. and the height of each cup is to be 4 in. Not allowing for overlapping or waste, find the number of square feet of tin that will be actually put into the cups.

7. A cylindric tank, open at the top and made of sheet iron $\frac{1}{4}$ in. thick weighing 450 lb./cu. ft., is 6 ft. deep and 7 ft. in diameter, inside measurements. Find the weight of the empty tank.

8. If the depth of the water in a circular well 5 ft. in diameter is 11 ft., how many cubic feet of water are there in the well? At 7.5 gal./cu. ft., how many gallons are there?

Pyramids. Another important geometric solid is the *pyramid*, — the solid which has a polygon for its base and triangles which meet at a common vertex for its lateral faces.

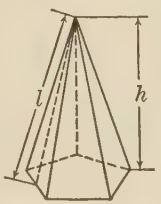


If the base of a pyramid is a regular polygon and the vertex of the pyramid is directly over the center of the base, the solid is called a *regular pyramid*. In this type, which is illustrated by the first of these figures, the height of any one of the triangular faces is called the *slant height* of the pyramid.

The second figure, above, illustrates an *oblique pyramid* with a triangular base. Pyramids are also classified as *triangular*, *quadrangular*, *pentagonal*, and so on, according as their bases are triangles, quadrilaterals, pentagons, and so on.

Since the lateral faces of an oblique pyramid are not congruent, we can apply the term "slant height" only to a regular pyramid.

Lateral Area of a Regular Pyramid. The lateral area of a regular pyramid is evidently the sum of the areas of the congruent triangular faces. From page 246 the area of each face is $\frac{1}{2}\,sl$, where s is the length of each side of the base and l is the slant height. If we now let n represent the number of sides in the base, it is evident that the lateral area L of the pyramid is given by the formula



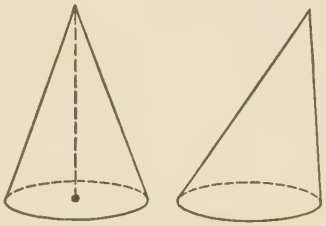
$$L = \tfrac{1}{2}\,nsl.$$

The formula may also be written as $L = \frac{1}{2}\,lp$, where p represents the perimeter of the base.

Volume of a Pyramid. It is proved in geometry that the volume of a pyramid is one third the volume of a prism having the same base and the same height. The formula for the volume of any pyramid, whether regular or oblique, is therefore

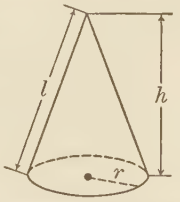
$$V = \tfrac{1}{3}\,Bh.$$

Cones. As in the case of the cylinder, the only type of cone that we shall need to consider is the *circular cone*, which is so called because its base is a circle. The first of the accompanying figures shows a *right circular cone*, which evidently corresponds to the regular pyramid shown on the opposite page, and the second figure shows an *oblique circular cone*, which corresponds to the oblique pyramid. The term *slant height* is applied to a right circular cone in the same way as to a regular pyramid.



Since a right circular cone can be generated by revolving a right triangle about one of the arms of the right angle as an axis, it is frequently called a *cone of revolution.*

Area of the Curve Surface of a Cone. If we consider a right circular cone in relation to a regular pyramid, we see that the perimeter ns, or p, of the base of the pyramid corresponds to the circumference C, or $2\pi r$, of the base of the cone. If we let l represent the slant height, the area S of the curve surface of the cone is therefore given by the formula



$$S = \pi rl.$$

The formula which would correspond to the pyramid formula $L = \frac{1}{2}\, lp$ would be $S = \frac{1}{2}\, Cl$, where C represents the circumference of the base.

Volume of a Cone. The cone bears the same relation to the cylinder as the pyramid does to the prism; that is, the volume of a cone is shown in geometry to be one third the volume of a cylinder having the same base and the same height. The formula for the volume of any circular cone, whether right or oblique, is therefore given by the formula

$$V = \tfrac{1}{3}\pi r^2 h.$$

The general formula $V = \frac{1}{3}\, Bh$ applies, of course, to any cone regardless of the shape of the base.

Exercises. Pyramids and Cones

1. Find the area of the total surface and the volume of a regular pentagonal pyramid inscribed in a circular cone 8 in. in diameter and 10 in. high.

2. Write a formula for finding the volume of a regular quadrangular pyramid with eight equal edges in terms of the length e of an edge.

3. Write a formula for the volume of a cone of revolution in terms of the slant height l and the radius r of the base.

4. Find the length of a side of the base of a regular hexagonal pyramid in which the volume is 249.4 cu. in. and the height is 8.0 in.

5. If the volume of a triangular pyramid is 240 cu. cm., and the sides of the base are 6 cm., 8 cm., and 10 cm., what is the height of the pyramid?

6. A gilt cone on top of a flagpole has a base with an area of $7\frac{1}{2}$ sq. in. and a height of 14 in. What is its volume?

7. An ice-cream cone is 4 in. deep and 2 in. across the top. How many cubic inches of ice cream does the cone hold when it is level full?

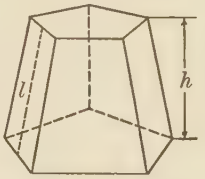
8. Find the volume of a regular pyramid whose slant height is 6.0 in. and whose base is an equilateral triangle inscribed in a circle 3.4 in. in diameter.

9. Find the area of the curve surface of a right circular cone which has a slant height of 10.2 in. and a base that is 22.8 in. in circumference.

10. Find the lateral area of a regular pyramid in which the base is an octagon 4 in. on a side and the slant height is 6 in.

11. If the curve surface of a right circular cone is unrolled, the resulting figure is a sector of a circle. Find the angle of the sector formed by unrolling a cone with a radius of 5.6 in. and a slant height of 12.4 in.

Frustum of a Pyramid. If we cut off the top of a pyramid by a plane parallel to the base, the part of the pyramid that is left is called a *frustum of the pyramid.*



In this figure, which shows a frustum of a regular pyramid, it is evident that the lateral faces are congruent trapezoids, and that the area of each one is $\frac{1}{2}\, l(s + s')$, where l represents the slant height and s and s' the sides of the lower and upper bases respectively. Letting n represent the number of sides, for the lateral area L we then have

$$L = \tfrac{1}{2}\, nl(s + s').$$

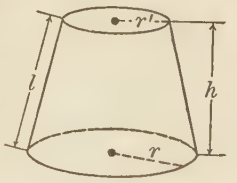
The formula for the lateral area may also be written as $L = \frac{1}{2}\, l(p + p')$, where p and p' represent the perimeters of the lower and upper bases.

It is proved in geometry that the formula for the volume V of a frustum of any pyramid, whether regular or oblique, is

$$V = \tfrac{1}{3}\, h(B + B' + \sqrt{BB'}),$$

where h represents the height of the frustum, and B and B' represent the areas of the lower and upper bases respectively.

Frustum of a Cone. From the figure below, which shows a frustum of a right circular cone, we see that the circumferences $2\,\pi r$ and $2\,\pi r'$ of the lower and upper bases correspond respectively to the perimeters ns (or p) and ns' (or p') of the bases of a frustum of a regular pyramid. For the area S of the curve surface of the conic frustum we thus have



$$S = \tfrac{1}{2}\, l(2\,\pi r + 2\,\pi r'),$$

or

$$S = \pi l(r + r').$$

Similarly, in finding the volume of the conic frustum, B and B' in the pyramid formula, above, may be replaced by πr^2 and $\pi r'^2$ respectively. We therefore have

$$V = \tfrac{1}{3}\, h\,(\pi r^2 + \pi r'^2 + \sqrt{\pi r^2 \cdot \pi r'^2}),$$

or

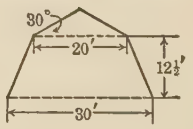
$$V = \tfrac{1}{3}\, \pi h(r^2 + r'^2 + rr').$$

Exercises. Measuring Frustums

1. How much earth was removed in making an excavation that is 6 ft. deep, 40 ft. square at the top, and 36 ft. square at the bottom?

2. The bottom and top diameters of a tub are to be 24 in. and 30 in. respectively. What depth should be allowed in order that the tub shall hold just 30 gal.? (1 gal. = 231 cu. in.)

3. This figure shows the cross section of the roof of a cylindric silo. The lower part of the roof is a frustum of a right circular cone with bases 30 ft. and 20 ft. in diameter and $12\frac{1}{2}$ ft. apart. The upper part of the roof is a cone whose curve surface makes an angle of 30° with the upper base of the frustum. Allowing 20% extra for waste in cutting, find the number of square feet of roofing material needed to cover the entire surface.



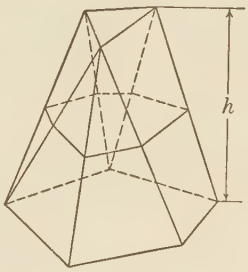
4. A Dutch stone windmill is in the shape of a frustum of a pyramid with square bases. The mill is 14 m. high, the outer dimensions at the bottom and the top are 18 m. and 14 m., and the inner dimensions are 14 m. and 12 m. respectively. How many cubic meters of stone were required to build it?

5. A right circular cone 10.5 in. high and 8.4 in. in diameter is cut by a plane that is parallel to the base at a distance of 4.8 in. from the vertex of the cone. Find the volume of the frustum thus formed.

6. A brick chimney has the shape of a frustum of a regular pyramid. The chimney is 160 ft. high, its upper and lower bases are squares 9 ft. and 14 ft. on a side respectively, and a square flue 6 ft. on a side runs from top to bottom. How much brickwork does the chimney contain?

7. What does the formula $V = \frac{1}{3}\pi h(r^2 + r'^2 + rr')$ on page 265 become when $r' = 0$? when $r = r'$? In each case explain the meaning of the result.

Volume of a Prismoid. It is occasionally necessary to find the volume of a *prismoid*, which is a solid having for its bases two polygons in parallel planes, and for its lateral faces triangles or trapezoids with one side common to one base and the opposite vertex or side common to the other base, as shown in this figure.



If we designate the volume by V, the area of the lower and upper bases by B and B' respectively, and the area of the mid-section (the section made by a plane parallel to the bases and halfway between them) by M, the formula for finding the volume of the prismoid is

$$V = \tfrac{1}{6}\, h(B + B' + 4\, M).$$

Exercises. The Prismoid

1. If the areas of the lower base, the mid-section, and the upper base of a prismoid are respectively 9 sq. in., 7 sq. in., and 5 sq. in., and the height is 4 in., what is the volume?

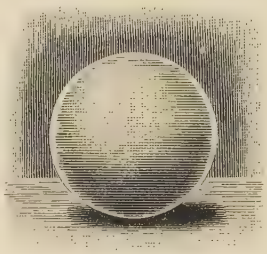
2. What is the volume of a prismoid 18 in. high in which the upper base is an isosceles right triangle with two equal sides of 12 in. and the lower base is a square 12 in. on a side?

3. Find the volume of a prismoid whose upper base is a square 16 in. on a side, whose lower base is a regular octagon 12 in. on a side, and whose height is 3 ft.

4. What is the volume of a blunt wedge that is 14 in. long, and whose ends are rectangles with dimensions of $\frac{1}{2}$ in. by 4 in. and 2 in. by 5 in. respectively?

5. In a concrete bridge pier the butt, which is 6 ft. high, has the shape of a prismoid. The lower base is an isosceles trapezoid with bases of 20 ft. and 16 ft. respectively and a height of 10 ft., and the upper base is a rectangle 6 ft. by 12 ft. Find the volume of concrete in the butt.

Area of a Sphere. The most common solid whose entire surface is a curve surface is the *sphere*. In finding the surface area S of a sphere when the radius r is known, we generally use the formula



$$S = 4\,\pi r^2.$$

Since $r = \frac{1}{2}\,d$, a more convenient formula for finding the surface area of a sphere when the diameter d is known is

$$S = \pi d^2.$$

The formula $S = 4\,\pi r^2$ reveals one interesting property of the sphere. That is, since πr^2 is the area of a great circle of the sphere (the circle formed by passing a plane through the center of the sphere), it shows that the area of the curve surface is four times that of a related plane surface.

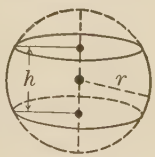
Volume of a Sphere. The formula used in finding the volume of a sphere when the radius r is known is

$$V = \tfrac{4}{3}\,\pi r^3,$$

and the similar formula used when the diameter d is known is

$$V = \tfrac{1}{6}\,\pi d^3.$$

Area of a Zone. A portion of a sphere included between two parallel planes, as shown in the figure below, is called a *zone*. We may think of a zone as the surface generated by an arc of a great circle revolving about the diameter of the circle as an axis.



On the earth we have what is known as the "torrid zone," which is that part of the earth's surface included between the planes of the tropics of Cancer and Capricorn.

If we let S represent the area of a zone on a sphere with the radius r, and h the height of the zone (the perpendicular distance between the planes), the formula for computing the area of the zone is

$$S = 2\,\pi r h.$$

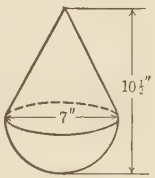
In the special case in which $h = 2\,r$, we have $S = 2\,\pi r \cdot 2\,r = 4\,\pi r^2$; that is, the zone covers the whole sphere.

Exercises. Measuring the Sphere

1. Find the least amount of wood that must be wasted in cutting a cube from a wooden sphere 4 in. in diameter.

2. Find the volume of the largest sphere that can be cut from a cone of revolution 14 in. high and 12 in. in diameter.

3. An iron casting with the shape shown in this figure is $10\frac{1}{2}$ in. long, and the base of the cone upon which the hemispherical base of the casting is constructed has a diameter of 7 in. Taking the weight of 1 cu. in. of cast iron as 0.26 lb., find to the nearest ounce the weight of the casting.



4. A brass sphere 1 in. in diameter is made of an alloy containing one part of zinc to two parts of copper by volume. Taking the weight of 1 cu. ft. of copper as 550 lb. and that of 1 cu. ft. of zinc as 428 lb., find the weight of the sphere.

5. If a cube and a sphere have equal volumes, what is the ratio of the radius of the sphere to the edge of the cube?

6. How many square feet of sheet copper are needed to cover a hemispherical roof which measures 35 ft. across?

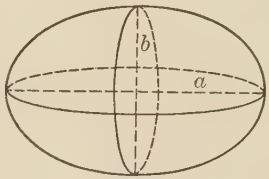
7. Find the diameter of a solid sphere made by melting and recasting the metal in two solid spheres of lead that are 2 in. and 3 in. respectively in diameter.

8. If without changing its volume a spherical drop of water is vaporized into a spray of 1000 equal spheres, by what per cent is the area of the surface of the drop increased?

9. Taking the earth as a sphere with a radius of 4000 mi., find to the nearest 1000 sq. mi. the area of the earth.

10. A sphere which has a diameter of 12 in. is cut from a solid lead cube 12 in. on an edge. If the pieces cut off are then melted and cast into another sphere, what is the diameter of this second sphere?

Ellipsoid of Revolution. Another solid in which the entire surface is a curve surface is called an *ellipsoid of revolution.* As shown in this figure, such a solid can be generated by revolving an ellipse about an axis.



If the ellipse is revolved about the minor axis, the solid is called an *oblate spheroid*; if it is revolved about the major axis, as in the figure, the solid is called a *prolate spheroid.*

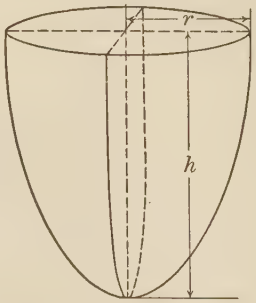
The formulas for the volume of these two types of ellipsoids in terms of the semiaxes a and b are as follows:

Oblate Spheroid, $V = \frac{4}{3}\pi a^2 b.$

Prolate Spheroid, $V = \frac{4}{3}\pi a b^2.$

It is evident that if the difference between a and b is comparatively small, the ellipsoid will be nearly spherical in shape.

Paraboloid of Revolution. This figure shows a *paraboloid of revolution,* the solid generated by revolving a parabola about its axis.



The base of a paraboloid is a circle, although in the figure it looks like an ellipse because it is seen obliquely.

The formula for the volume of this solid in terms of the height h and the radius r of the base is

$$V = \frac{1}{2}\pi r^2 h.$$

This type of solid is occasionally used in designing tanks for various purposes, and the inner surface is frequently used as the reflecting surface in a parabolic mirror.

Similar Solids. It is proved in geometry that in similar solids (that is, in solids which have the same shape but not necessarily the same size) the areas of corresponding surfaces are proportional to the squares of corresponding dimensions, and the volumes (or corresponding portions thereof) are proportional to the cubes of corresponding dimensions.

Exercises. Ellipsoids and Paraboloids

1. What is the difference between the volume of a sphere 12 in. in diameter and that of an oblate spheroid having a polar diameter of 10 in. and an equatorial diameter of 14 in.?

2. A regulation football has the shape of a prolate spheroid. Taking the longer axis as $10\frac{1}{2}$ in. and the shorter axis as $7\frac{1}{4}$ in., find the volume of the ball.

3. A parabolic reflector is 20 in. deep, and the diameter at the outer end is 35 in., inside measurements. What is the number of cubic inches of air space between the reflector and a flat lens placed over the outer end?

4. The earth is flattened at the poles so that its shape approximates that of an oblate spheroid. Taking the polar diameter as 7900 mi. and the equatorial diameter as 7926 mi., find the volume of the earth. What is the per cent of error in computing the volume of the earth as a sphere with a radius of 4000 mi.?

5. When liquid is placed in a rotating cylindric container, it tends to hug the walls of the container, the inner surface of the liquid taking the form of a paraboloid of revolution. Find the volume of milk in a cream separator in which the rate of rotation of the cylindric milk container, which is 6 in. in diameter and $4\frac{1}{2}$ in. high, is sufficient to cause the upper rim of the milk to reach the upper edge of the container when the depth of the milk at the center of the base is 1 in.

6. If the major axis of an ellipse is 12 in. long and the minor axis is 8 in. long, what is the difference between the volumes of the two solids formed by the successive revolution of the ellipse about its major and minor axes respectively?

7. If the semiaxes of an ellipsoid of revolution are equal, what is the shape of the figure? That is, in the special case in which $a = b$, to what familiar formula do the two formulas for the volume of an ellipsoid reduce?

Exercises. Miscellaneous Problems

1. A covered box, which is made of steel $\frac{1}{8}$ in. thick weighing 490 lb./cu. ft., is 16 in. long, 9 in. wide, and 4 in. deep, inside measurements. What is the weight of the box?

2. Find the weight of a bar 6 ft. long, 2 in. wide, and $1\frac{3}{4}$ in. thick made of steel weighing 490 lb./cu. ft.

3. Find the thickness of a sheet of gold leaf made by beating out an inch cube of gold so that the sheet will cover an area of 25,000 sq. in.

4. At 54 lb./cu. ft., find the weight of a wooden beam that is 8 in. by 10 in., and 18 ft. long.

5. In a rectangular irrigation ditch 3 ft. wide the water, which is 1 ft. 6 in. deep, is flowing at the rate of 8 ft./min. Allowing $7\frac{1}{2}$ gal./cu. ft., find the amount of water which flows past a given point in 1 hr.

6. A rectangular water tank is 9 ft. long and 5 ft. 6 in. wide, inside measurements. How much water must be drawn off in order to lower the surface of the water 1 ft.?

7. Each dimension of a rectangular container that holds exactly 1 gal. (231 cu. in.) is an integral number of inches in length. What are the dimensions of the container?

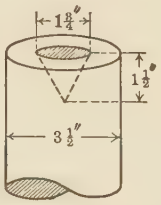
8. When a cube of iron 6 in. on an edge is melted, it just fills a rectangular mold whose base is 8 in. long and 6 in. wide. Find the depth of the mold and the difference between the total surface area of the new casting and that of the cube.

9. If it is desired to double the capacity of an upright cylindric boiler but to keep the diameter and the height in their original ratio, by what per cent will the area of the total surface of the boiler be increased?

10. A cylindric hollow column with an outside diameter of 6 in. and 12 ft. long is made of cast iron 1 in. thick weighing 450 lb./cu. ft. What is the weight of the column?

11. During a rainfall of $\frac{1}{2}$ in., to what depth will a circular well 5 ft. 6 in. in diameter be filled by the drainage from a rectangular court which is 30 ft. long and 26 ft. 6 in. wide?

12. A conic depression is cut into each end of a cylindric iron roller 36 in. long, one end of which is here shown. Find the number of cubic inches of metal in the roller.



13. Solve Ex. 12, supposing that each depression is cut only half as deep.

14. Find the volume of a cone that has the same base and height as a cylinder 12 in. high and 6.3 in. in diameter.

15. If the volume of a triangular pyramid is 350 cu. in. and the sides of the base are 5 in., 6 in., and 8 in., what is the height of the pyramid?

16. A pump has a cylindric barrel 4 in. in diameter, and the volume of water in 1 ft. of the length is pumped at each stroke. Find the number of strokes necessary to fill a tub in the form of a frustum of a cone 36 in. in diameter at the bottom, 48 in. in diameter at the top, and 18 in. deep.

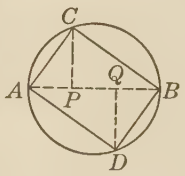
17. A horizontal cylindric tank which is 5.5 ft. in diameter and 18 ft. long is filled with gasoline to a depth of 20 in. Allowing $7\frac{1}{2}$ gal./cu. ft., find the amount of gasoline in the tank.

18. A room is heated by 265 ft. of steam pipe having an outside diameter of $2\frac{5}{8}$ in. Find to the nearest 0.1 sq. ft. the area of the radiating surface.

19. Find the cost, at \$1.75/sq. ft., of polishing the curve surface of a monument in the shape of a conic frustum in which the slant height is 12 ft. and the radii of the lower and upper bases are 3 ft. 6 in. and 2 ft. 4 in. respectively.

20. The length of each side of a triangular plate of wrought iron $\frac{3}{4}$ in. thick is 3 ft. 4 in. If the weight of a plate of this metal 1 ft. square and $\frac{1}{8}$ in. thick is 5 lb., what, to the nearest 1 lb., is the weight of the triangular plate?

21. A convenient rule for calculating the strongest beam that can be cut from a cylindric log is as follows: Trisect the diameter AB, and at the points of division P and Q draw perpendiculars meeting the circle at C and D respectively on opposite sides of AB. The rectangle $ADBC$ is a cross section of the required beam. Using this rule, find the dimensions of the strongest beam that can be cut from a log which is 18 in. in diameter; from one 20 in. in diameter.



22. In making a cylindric container 18 in. in diameter that will hold 60 gal. of water (1 gal. = 231 cu. in.), what must be the height of the cylinder to the nearest 0.1 in.?

23. A hollow steel shaft which is 14 ft. long has an outside diameter of 16 in. and an inside diameter of 9 in. Allowing 0.29 lb./cu. in., find the weight to the nearest pound.

24. At 100 lb./cu. ft., what is the weight of a concrete slab 8 ft. long, 3 ft. wide, and 6 in. thick?

25. Taking the weight of 1 cu. ft. of steel as 490 lb., find the weight of a 12-foot bar of steel having a cross section that is a regular hexagon with a diameter of $\frac{5}{8}$ in.

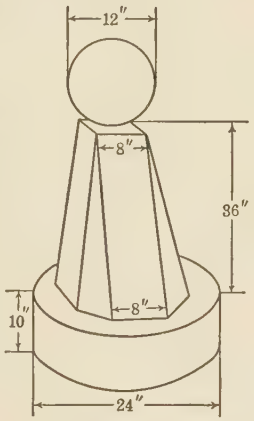
26. Find the area of the curve surface of a tapered rod 6 in. long in which the smaller diameter is 0.250 in. and the larger diameter is 0.625 in. If the rod is made of steel which weighs 490 lb./cu. ft., what is its weight?

27. If the bottom of a bushel basket (1 bu. = 2150 cu. in.) is a hemisphere 15 in. in diameter, what, to the nearest inch, is the height of the cylindric upper portion? What would be the proper dimensions for a half-bushel basket of the same shape?

28. A concrete smokestack 120 ft. high has a uniform inside diameter of 4 ft., the outside diameter at the base is 16 ft., and the outside diameter at the top is 8 ft. Disregarding the volume of the reënforcing material, find the number of cubic yards of concrete in the stack.

29. Find the area of the reflecting surface of a white porcelain lamp shade which has the shape of a frustum of a cone in which the diameter of the upper base is 3 in., that of the lower base is 14 in., and the height is 4 in.

30. In the monument shown in this figure find the volume of the ball mounted at the top.



31. In Ex. 30 find the volume of the cylindric base of the monument.

32. In Ex. 30 find the area of the base of the central portion which has the shape of a prismoid having a regular octagonal base 8 in. on a side.

33. In Ex. 30 find the volume of the prismoid that forms the central portion.

34. From Exs. 30–33 find the number of cubic yards of concrete necessary to make the monument.

35. When water is flowing through an 8-inch drain tile at a depth of 6 in. with a velocity of 1.4 ft./sec., what depth of rainfall will it remove from a 30-acre field in 24 hr.? If the water fills the drain tile and flows at the same rate, from how many acres will it draw $\frac{1}{4}$ in. of rainfall in 24 hr.?

36. Taking the weight of settled hay in a cube 7 ft. on an edge as 1 T., find the number of tons in a round stack of hay whose ground diameter is 16 ft. and whose diameter 6 ft. from the ground is 18 ft., the portion above this point being a cone whose peak is 24 ft. above the ground.

37. Considering the earth as a sphere in which the radius is 4000 mi., find the area of the north temperate zone lying between the arctic circle, which is 23.5° from the north pole, and the tropic of Cancer, which is 23.5° from the equator.

38. Find the diameter of the top of a gallon measure which is 6 in. across the circular base and 9 in. high.

Exercises. Test on Chapter XII

Copy the right-hand column in Exs. 1–13 below, and insert in each square the number (the first one is 9) of the corresponding description given in the left-hand column:

1. Volume of a rectangular solid.	$S = \pi rl.$
2. Lateral area of a prism.	$L = \frac{1}{2} nsl.$
3. Area of curve surface of a cylinder.	$S = 4\pi r^2.$
4. Volume of a prism or parallelepiped.	$V = lwh.$
5. Area of curve surface of a conic frustum.	$V = \pi r^2 h.$
6. Lateral area of a regular pyramid.	$E = \pi ab.$
7. Volume of a sphere.	$S = 2\pi rh.$
8. Volume of a pyramid.	$V = \frac{1}{3}\pi r^2 h.$
9. Area of curve surface of a circular cone.	$S = \pi l(r + r').$
10. Volume of a circular cylinder.	$V = \frac{4}{3}\pi r^3.$
11. Volume of a circular cone.	$L = hp.$
12. Area of a sphere.	$V = Bh.$
13. Area of an ellipse.	$V = \frac{1}{3} Bh.$

14. A right circular cone 9.8 in. high with a base 6.3 in. in diameter is cut by a plane parallel to the base at a distance of 3.6 in. from the vertex. Find the area of the curve surface of the frustum thus formed.

15. Allowing 25% extra for waste, find the amount of leather needed to cover a basket ball 10 in. in diameter.

16. Find the difference between the volumes of two cubes respectively inscribed in and circumscribed about a sphere which is 2 in. in diameter.

17. In a grain bin, which is 8 ft. wide and 16 ft. long, the grain stands level 6 ft. deep for a length of 4 ft. at the back of the bin and then slopes uniformly to the floor at the front. Allowing $1\frac{1}{4}$ cu. ft./bu., find the number of bushels of grain which the bin contains.

Exercises. General Review

1. If to 10 gal. of a mixture of alcohol and water a certain amount of water is added, the amount of alcohol is 30% of the resulting mixture. If twice as much water had been added, the alcohol would have been only 20% of the mixture. How much water was actually added, and how much alcohol is there in the final mixture?

2. Draw the graph of the function $x^2 - x + 7$ and show how it tells us that there are no real roots for the equation $x^2 - x + 7 = 0$.

3. A flywheel 5.4 ft. in diameter is revolving at a speed of 120 R.P.M. (revolutions per minute). Find to the nearest 0.1 ft. the speed in feet per second at which a point on the rim of the wheel is traveling.

4. By means of logarithms find the value of $\sqrt[3]{\frac{22.7 \times 18.6}{2.5 \times 3.142}}$.

5. If y varies inversely as $\sqrt{x}$, and if $y = 8$ when $x = 9$, what is the value of x when $y = 16$?

6. On the first day of each month a man deposits D dollars in a bank where each deposit draws interest at the rate m per month, compounded monthly. What is the amount of his account at the end of k months?

7. What annual sinking-fund payment will provide for the payment of \$800,000 due in 30 yr., if the sinking fund accumulates at 4%, compounded annually?

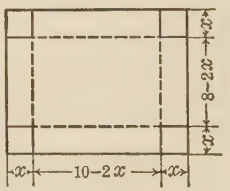
8. From two points, A and B, which are $\frac{3}{4}$ mi. apart on a main highway, two branch roads run to a point C that is 2560 ft. from A and 2730 ft. from B. At what angle do the two branch roads meet each other?

9. How many linear feet of brass wire 0.01 in. in diameter can be made by drawing out 1 cu. ft. of brass?

10. Write a formula for the volume V of a hollow spherical shell with the outside diameter d and the inside diameter d'.

11. A 12-inch sphere is placed inside a cubic box that is exactly 12 in. on an edge, inside measurements. How much space in the box is not occupied by the sphere? If the 12-inch sphere is now removed and replaced by 3-inch spheres, how many of the latter can be put into the box? How will the amount of space not occupied by the 3-inch spheres compare with the amount of space not occupied by the 12-inch sphere? Which is larger and by how much, the surface area of the 12-inch sphere or the combined surface area of the 3-inch spheres?

12. An open box is to be made from a rectangular sheet of tin by cutting equal squares from the corner, as here shown, and folding up the flaps. If the piece of tin measures 8 in. by 10 in., approximately how large a square should be cut from each corner so as to have the box contain 49 cu. in.?



13. The radii of the bases of a frustum of a right circular cone are 11 in. and 12.5 in. respectively, and the height of the frustum is 5 in. Find the length of the cone from which the frustum was originally cut, and find also the area of the curve surface and the volume of the cone.

14. A mechanic's handbook gives the relation between the standard size of a bolt and the bolthead by means of the formula $f = 1.5\,d + 0.125$. If the mechanic knows the value of f, how can he find the value of d?

15. If 1 cu. ft. of copper is drawn out into 5000 ft. of wire, what is the diameter of the wire to the nearest 0.01 in.?

16. If the ocean area is approximately three fourths of the earth's surface and the average depth of the water is 2 mi., what is the approximate volume of the water in the oceans?

17. If a cube and a sphere have equal volumes, what is the ratio of the radius of the sphere to the edge of the cube?

CHAPTER XIII

FORMULAS IN AGRICULTURAL ENGINEERING

Nature of the Work. In this chapter we shall consider several practical formulas that are important in agricultural engineering. Although the results from these formulas are generally approximate, since many of the factors involved are determined by experiment or by estimate, they are sufficiently accurate for any practical uses.

Concrete Mixtures. In high-grade concrete the sand with its voids fills the voids in the gravel or crushed stone, and the cement fills the remaining voids, so that the volume of the mixture is much less than the total volume of the ingredients.

The average allowance for voids is 45% for gravel or crushed stone and $33\frac{1}{3}$% for sand. Letting C, S, and G represent respectively the number of cubic yards of cement, sand, and gravel (or crushed stone) per cubic yard of mixture, we have

$$C + 0.66\tfrac{2}{3}\,S + 0.55\,G = 1. \qquad (1)$$

For a 1–3–5 mixture of concrete, which means 1 part cement, 3 parts sand, and 5 parts gravel (or crushed stone), it is evident that $S = 3\,C$ and $G = 5\,C$. From (1) we thus have

$$C + 0.66\tfrac{2}{3} \cdot 3\,C + 0.55 \cdot 5\,C = 1; \qquad (2)$$

whence $C = 0.174$, $S = 0.522$, and $G = 0.870$.

That is, 1–3–5 concrete requires 0.174 cu. yd. of cement, 0.522 cu. yd. of sand, and 0.870 cu. yd. of gravel (or crushed stone) for each cubic yard of mixture.

For example, for 200 cu. yd. of 1–3–5 mixture we should need 34.8 cu. yd. of cement, 104.4 cu. yd. of sand, and 174 cu. yd. of gravel.

From equation (1) we may develop equation (2) for any other mixture, for example, a 1–2–4 mixture, by proceeding in a similar manner.

Fuller's Formulas. The method outlined on page 279 gives the amount of cement in cubic yards. Cement, however, is more commonly measured by the barrel or by the sack containing $\frac{1}{4}$ bbl. Although a sack is frequently estimated as being equivalent to 1 cu. ft., it is more exact to take 3.8 cu. ft./bbl.

That is, to find the number of barrels of cement needed in the case on page 279 we might multiply the result in cubic yards by $27 \div 3.8$, or 7.11.

The following practical formulas, which are known as *Fuller's Formulas*, give for each cubic yard of mixture the required amount of cement (C) in barrels and the amounts of sand (S) and gravel (G) in cubic yards, as before:

$$C = \frac{11}{c+s+g} \qquad S = \frac{3.8\,Cs}{27} \qquad G = \frac{3.8\,Cg}{27}$$

The letters c, s, and g represent the number of parts of cement, sand, and gravel respectively in the mixture.

For example, if we are using a 1–2–4 mixture of concrete, we have

$$C = \frac{11}{1+2+4} = \frac{11}{7} = 1.57,$$

$$S = \frac{3.8 \cdot 1.57 \cdot 2}{27} = \frac{11.932}{27} = 0.44,$$

$$G = \frac{3.8 \cdot 1.57 \cdot 4}{27} = \frac{23.864}{27} = 0.88.$$

That is, for each cubic yard of 1–2–4 mixture it requires 1.57 bbl. (or 6.3 sacks) of cement, 0.44 cu. yd. of sand, and 0.88 cu. yd. of gravel.

Exercises. Concrete Mixtures

Find by Fuller's Formulas the amount of ingredients needed per cubic yard for the following concrete mixtures:

1. 1–1–2.
2. 1–2–5.
3. 1–2$\frac{1}{2}$–5.
4. 1–2–4$\frac{1}{2}$.
5. 1–3–6.
6. 1–3$\frac{1}{2}$–7.
7. 1–1$\frac{1}{2}$–3.
8. 1–2–3.
9. 1–4–8.

10. Solve Exs. 1–9 by the method given on page 279 and state in each case the differences between the results.

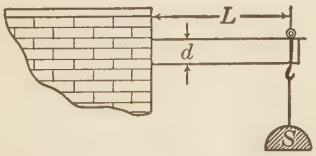
Safe Loads for Wooden Beams. Another problem of importance in agricultural engineering relates to the loading of beams. In our work we shall consider four simple types, the cases being classified according to the methods of supporting the beam and distributing the load. In the formulas we shall use the following scheme of notation:

b = breadth (thickness) of beam in inches
d = depth of beam in inches
k = coefficient of strength
L = length of the free span in feet
S = safe load in pounds

The value of the coefficient k, which depends upon the material of which the beam is composed, has been determined by experiment to be as follows: yellow pine and Douglas fir, 100; white pine, 60; hemlock, 50; white oak, 75; chestnut, 55; cast iron, 167; steel, 889; limestone, 8.

Case 1. In the first type the beam is supported at one end only, and the load is concentrated at the other. The formula for the safe load in this case is

$$S = \frac{kbd^2}{4L}.$$



For example, a farmer wishes a hay-fork beam 4 ft. long, supported at one end, that will carry a load of 1000 lb. concentrated at the other end. If he uses white pine 3 in. thick, what depth, to the next larger inch, should he allow for the beam?

Unless it is otherwise stated, the weight of the beam is to be disregarded. From the formula for the safe load S, given above, we then have

$$d^2 = \frac{4LS}{kb};$$

whence

$$d = 2\sqrt{\frac{LS}{kb}}$$

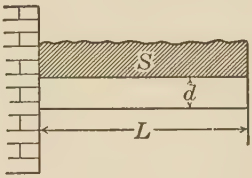
$$= 2\sqrt{\frac{4 \cdot 1000}{60 \cdot 3}}$$

$$= 9.42 \cdots.$$

That is, the beam should have a depth of 10 in.

Case 2. In this type the beam is supported at one end only, and the load is uniformly distributed along the beam. The formula for the safe load is

$$S = \frac{kbd^2}{2L}.$$



For example, a balcony, which is 12 ft. long and which overhangs 6 ft., is supported by three yellow-pine 6 × 10 sills turned on edge. Disregarding the weight of the balcony floor, what safe load per square foot of floor space will the balcony carry?

From the formula, the safe load for each sill is first found as follows:

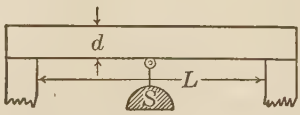
$$S = \frac{kbd^2}{2L} = \frac{100 \cdot 6 \cdot 10^2}{2 \cdot 6} = 5000.$$

For the three sills the load is 3 × 5000 lb., or 15,000 lb.

Since there are 72 sq. ft. of floor space, the safe load for the balcony is therefore $\frac{1}{72}$ of 15,000 lb., or 208 lb./sq. ft., to the nearest pound.

Case 3. In the third type, which applies to beams supported at both ends and loaded at the center, the formula for the safe load upon the beam is

$$S = \frac{kbd^2}{L}.$$



For example, find the stock depth of a yellow-pine beam 3 in. thick that will support a load of 1200 lb. at the center of a 12-foot span.

From the formula for the safe load S, given above, we have

$$d^2 = \frac{LS}{kb};$$

whence

$$d = \sqrt{\frac{LS}{kb}}$$

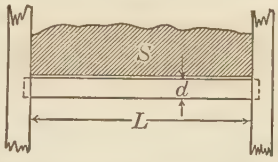
$$= \sqrt{\frac{12 \cdot 1200}{100 \cdot 3}}$$

$$= 6.92 \cdots.$$

That is, if stock were cut in odd-inch depths, it would be necessary to have a beam 7 in. deep. In practice, however, the stock would probably be cut only to even inches, and it would therefore take a 3 × 8 beam.

Case 4. In the fourth and last type, which applies to beams supported at both ends and loaded uniformly along the span, the formula for the safe load is

$$S = \frac{2\,kbd^2}{L}.$$



For example, the floor of a barn loft 40 ft. long is supported by 3 × 8 yellow-pine timbers having a span of 12 ft. and placed edgewise 2 ft. apart between centers. Allowing 14 lb./sq. ft. for the weight of the floor, find the number of tons of hay, distributed uniformly, that can be safely stowed in the loft.

For the safe load that can be put on each floor timber we first have

$$S = \frac{2\,kbd^2}{L} = \frac{2 \cdot 100 \cdot 3 \cdot 8^2}{12} = 3200.$$

The number of timbers is $\frac{1}{2} \cdot 40 + 1$, or 21, and hence the total safe load for the timbers is 21 · 3200 lb., or 67,200 lb.

The total floor load is 12 · 40 · 14 lb., or 6720 lb., and hence for the number of tons of hay we have

$$\frac{67{,}200 - 6720}{2000} = 30.24.$$

That is, approximately 30 T. of hay, if uniformly distributed over the floor, can be safely stowed in the loft.

Stock Dimensions. Lumber is cut in stock depths (values of d) of 1 in., 2 in., 3 in., 4 in., 6 in., 8 in., and so on up to 24 in. Depths in excess of 16 in., however, are uncommon and are expensive. For depths up to 12 in. stock thicknesses (values of b) are 1 in., 2 in., and 3 in., and for depths from 4 in. up the thicknesses run in even inches (4 in., 6 in., 8 in., · · ·).

Although from the formulas it would appear that the greater the value of d in comparison with b, the greater the economy of material, the thickness gives lateral stiffness which prevents the beam from buckling. In selecting stock dimensions that will insure a safe lateral stiffness for a beam, good practice recommends a thickness not less than $\frac{1}{4}\,d$, although $\frac{1}{6}\,d$ is sometimes taken as the lower limit.

Exercises. Loads for Beams

1. Find the stock depth of a 10-inch white-oak beam 12 ft. long, supported at both ends, that will safely carry a concentrated center load of 6 T.

2. What uniformly distributed safe load can be carried by a 3×12 Douglas-fir plank 12 ft. long, turned flat, and supported at both ends? If the plank is turned edgewise, by how much can the safe load be increased?

3. A 6×8 yellow-pine timber 16 ft. long, turned edgewise, is used as a lever to aid in lifting the corner of a house to permit making alterations in the foundation. If the fulcrum is placed 4 ft. from the end which is put under the house, what safe pressure may be exerted on the outer end of the timber?

4. A 12-inch white-pine plank, placed flat, which is to be used as a springboard, is to project 8 ft. beyond its support. How thick must the plank be in order to bear safely the weight of four men weighing 150 lb. each and standing at equal distances along the projecting portion of the plank?

5. Compare the strength of a 2×8 Douglas-fir plank 14 ft. long, turned edgewise and supported at both ends, with the strength of the same plank similarly supported but turned flat.

6. Which is the stronger beam, a 3×10 plank turned edgewise or a 6×8 timber turned flatwise? What is the difference in the strength of the two beams?

7. If a hay-fork beam must carry a load of 1200 lb. concentrated at a point 4 ft. from the single support, what are the most economical stock dimensions for a white-oak timber? What other dimensions might be used?

Since in this case $S = 1200$, $k = 75$, and $L = 4$, we have $1200 = \frac{75\,bd^2}{4 \cdot 4}$. Although both b and d are unknown, we may solve for b in terms of d and then assume convenient values for d. After computing the corresponding values of b, we may then select a pair of dimensions that will meet the conditions stated in the last paragraph on page 283.

8. A 2-inch hemlock plank floor is supported by yellow-pine timbers with a span of 16 ft. and spaced 16 in. between centers. If, in addition to the weight of the floor and the weight of the timbers, the floor is to carry a uniformly distributed load of 125 lb./sq. ft., what are the most economical stock dimensions for the timbers?

For the weight per cubic foot of construction materials consult page 349.

9. What stock dimensions should be allowed for two Douglas-fir timbers, 40 ft. long between supports, in order that they may safely carry, in addition to their own weight, a 5-ton car running on tracks made of two rails each weighing 75 lb./yd.?

10. What load, concentrated at a point 6 ft. from the support, can be safely carried by a 6×8 Douglas-fir beam placed edgewise and supported at one end?

11. In Ex. 10 the weight of the beam (see page 349) may be considered as a uniformly distributed load. Taking this into consideration, determine the depth of the beam needed to carry the same concentrated load.

Add half the uniformly distributed load to the concentrated load.

12. In each of the four types of beams mentioned on pages 281–283, what effect upon the safe load is produced by doubling the breadth of the beam? by doubling the depth?

13. An unloading platform 6 ft. wide along the front wall of a freight shed 60 ft. long has a 3-inch plank floor of Douglas fir resting upon 3×12 fir timbers placed edgewise and spaced 16 in. between centers. The timbers project the full width of the platform and are supported only at the wall end. What is the total uniformly distributed load, in excess of the weight of the platform, that can be safely carried by the timbers?

14. Suppose that in Ex. 13 it is desired to double the safe load that can be carried by the platform by placing fir beams supported every 12 ft. beneath the outer edge of the platform. What dimensions should these beams have?

Flow of Water in Pipes. A third important problem in agricultural engineering relates to the flow of water in pipes. In our work we shall use the following scheme of notation:

A = area of cross section of pipe in square feet
d = diameter of pipe in feet
f = coefficient of friction
g = acceleration of gravity
h = hydrostatic head in feet
L = length of pipe in feet
q = discharge in cubic feet per second
Q = discharge in gallons per hour
v = velocity of flow in feet per second

Disregarding the question of friction, the theoretical formula for the velocity of flow is

$$v = \sqrt{2gh}. \tag{1}$$

The value of g may be taken as 32.2 ft. per second per second, and hence for practical use a more convenient formula for v is $v = 8.02\sqrt{h}$.

The velocity of flow, however, is so greatly reduced by friction, which is due to various causes and particularly to the length of the pipe, that for practical purposes it is necessary to use formulas allowing for friction. For short pipes, where $L \leqq 500\,d$, the formula, according to Merriman, is

$$v = \sqrt{\frac{2ghd}{1.5\,d + fL}}, \tag{2}$$

while for long pipes, where $L > 500\,d$, the formula is

$$v = \sqrt{\frac{2ghd}{fL}}. \tag{3}$$

Taking g as above, these formulas may be more conveniently written as

$$v = 8.02\sqrt{\frac{hd}{1.5\,d + fL}} \quad \text{and} \quad v = 8.02\sqrt{\frac{hd}{fL}}.$$

The factor 1.5 in the denominator provides for friction which is due to the entrance and to the constriction of the cross-section area at outlets. Since this element is relatively small in comparison with the effects of friction due to the length of the pipe, it is disregarded for long pipes.

Determining the Value of v. Since problems relating to the flow of water in pipes always lack some of the necessary data, we have to assume at first approximate values of the missing factors. From these assumed values a test computation is then made, and from the results of this computation new and more accurate values of the missing factors are assumed. The computation is then repeated, and the process is continued until a final result is obtained which fulfils the conditions of the problem within practical limits.

For example, to find the velocity of flow v, we need to know the value of the friction coefficient f, which depends upon v.

On page 350 will be found a table giving for standard sizes of pipe the values of f corresponding to different values of v. For values of v lying between those given in the table, ordinary interpolation should be used.

In order to find an approximate value of v from which to find the value of f from the table, we first compute the theoretical value of v by means of formula (1) on page 286. We then take 25% to 30% of this value of v in a case dealing with short pipes, or 5% to 10% in a case dealing with long pipes, and find the corresponding value of f from the table. Using this value of f, we then compute v more exactly by formula (2) for short pipes or by formula (3) for long pipes. This result is then used to obtain a new and more exact value of f, and the value of v is recomputed. This process is repeated until we can find two consecutive values of v which have a difference so small as to be negligible for ordinary purposes. In practical hydraulics, results to more than two decimal places are rarely necessary.

Formulas for the Discharge. After finding the value of v, as described above, we may use the following formulas for q, the discharge in cubic feet per second, and Q, the discharge in gallons per hour:

$$q = Av \tag{4}$$

$$Q = 7.48 \cdot 60^2 \cdot Av \tag{5}$$

In the formula for Q the factor 7.48 is the number of gallons per cubic foot, while the factor 60^2 is obviously the number of seconds in 1 hr.

Illustrative Example for Short Pipes. Compute to the nearest 10 gal. the discharge per hour under a head of 25 ft. that can be expected from a $\frac{3}{4}$-inch pipe 30 ft. long.

Following the procedure outlined on page 287, we first find the theoretical value of v. From formula (1) on page 286 we have

$$\begin{aligned} v &= 8.02\sqrt{h} \\ &= 8.02\sqrt{25} \\ &= 40.10. \end{aligned}$$

Since $d = \frac{\frac{3}{4}}{12} = \frac{1}{16}$, it is evident that $L < 500\,d$, and hence for the value of v from which to determine the first assumed value of f we have

$$0.25 \times 40.10 = 10, \text{ approximately.}$$

For a $\frac{3}{4}$-inch pipe we find that, when $v = 10$, the table of friction coefficients on page 350 gives the value of f as 0.028.

For the first trial computation of the value of v we therefore have

$$\begin{aligned} v &= 8.02\sqrt{\frac{hd}{1.5\,d + fL}} \\ &= 8.02\sqrt{\frac{25 \cdot \frac{1}{16}}{1.5 \cdot \frac{1}{16} + 0.028 \cdot 30}} \\ &= \frac{40.10}{\sqrt{14.940}} \\ &= 10.38, \text{ approximately.} \end{aligned}$$

For $v = 10.38$ the table on page 350 gives the value of f by interpolation as 0.0279, and hence for the second trial computation we have

$$\begin{aligned} v &= 8.02\sqrt{\frac{25 \cdot \frac{1}{16}}{1.5 \cdot \frac{1}{16} + 0.0279 \cdot 30}} \\ &= \frac{40.10}{\sqrt{14.89}} \\ &= 10.39, \text{ approximately.} \end{aligned}$$

This value of v does not change f within the fourth decimal place, and hence we need not continue further the process of computing v.

Since $A = \frac{1}{4}\pi d^2$,

we have $A = 0.7854(\frac{1}{16})^2 = 0.0031.$

From formula (5), which is given on page 287, we therefore have

$$\begin{aligned} Q &= 7.48 \cdot 60^2 \cdot Av \\ &= 7.48 \cdot 3600 \cdot 0.0031 \cdot 10.39 \\ &= 867.2, \text{ approximately.} \end{aligned}$$

That is, to the nearest 10 gal., the expected discharge is 870 gal./hr.

Illustrative Example for Long Pipes. Determine the diameter of the pipe that, under a head of 100 ft., will deliver 1000 bbl. of water per day through a length of 3600 ft.

Using the three equivalents 1 bbl. = 31.5 gal., 1 gal. = 231 cu. in., and 1 cu. ft. = 1728 cu. in., we first find that for the discharge q in cubic feet per second we have

$$q = \frac{1000 \cdot 31.5 \cdot 231}{1728 \cdot 60 \cdot 60 \cdot 24} = 0.0487.$$

For trial, we may first assume that a $1\frac{1}{2}$-inch pipe will meet the given conditions, and for such a pipe we have

$$d = \frac{\frac{3}{2}}{12} = \frac{1}{8},$$

and $$A = \tfrac{1}{4}\,\pi d^2 = 0.7854(\tfrac{1}{8})^2 = 0.0123.$$

Since $$q = Av,$$

we have $$v = \frac{q}{A} = \frac{0.0487}{0.0123} = 4, \text{ approximately.}$$

For a $1\frac{1}{2}$-inch pipe we find from the table on page 350 that $f = 0.028$ when $v = 4$.

Then since $$v = 8.02\sqrt{\frac{hd}{fL}},$$

we have $$q = 8.02\,A\sqrt{\frac{hd}{fL}}$$

$$= 8.02 \cdot 0.0123\sqrt{\frac{100 \cdot \frac{1}{8}}{0.028 \cdot 3600}}$$

$$= 0.03474.$$

Since this value of q is less than the required discharge, we must try the next larger stock size of pipe, which is a 2-inch pipe. In this case

$$d = \tfrac{2}{12} = \tfrac{1}{6},$$

and $$A = \tfrac{1}{4}\,\pi d^2 = 0.7854(\tfrac{1}{6})^2 = 0.0218.$$

Computing the value of v as before, we then have

$$v = \frac{q}{A} = \frac{0.0487}{0.0218} = 2.23, \text{ approximately.}$$

For a 2-inch pipe we find from the table on page 350 that $f = 0.0295$ when $v = 2.23$, and hence, as before, we have

$$q = 8.02 \cdot 0.0218\sqrt{\frac{100 \cdot \frac{1}{6}}{0.0295 \cdot 3600}}$$

$$= 0.06928.$$

Although this value of q is greater than the specified discharge, it is evident that it will be necessary to use a 2-inch pipe.

Exercises. Flow of Water in Pipes

1. Under a head of 16 ft., what will be the velocity of flow in a 200-foot length of pipe $1\frac{1}{2}$ in. in diameter?

2. In Ex. 1, what will be the velocity of flow under a head of 8 ft.? a head of 32 ft.?

3. In Ex. 1, what will be the velocity of flow for a length of 100 ft.? for a length of 400 ft.?

4. In Ex. 1, what will be the velocity of flow for a pipe $\frac{3}{4}$ in. in diameter? for a pipe 3 in. in diameter?

5. Under a pressure of 10 lb./sq. in., what will be the velocity of flow from a 1-inch pipe 30 ft. long?

A pressure of 1 lb./sq. in. is equivalent to a head of 2.304 ft., and a head of 1 ft. is equivalent to a pressure of 0.434 lb./sq. in.

6. In Ex. 5, what will be the velocity of flow under a pressure of 5 lb./sq. in.?

7. In Ex. 5, what will be the velocity of flow for a 20-foot length of pipe? for a 60-foot length of pipe?

8. In Ex. 5, what will be the effect upon the discharge if the 1-inch pipe is replaced by a $\frac{3}{4}$-inch pipe? if it is replaced by a $1\frac{1}{2}$-inch pipe?

9. In long pipes, what is the effect upon the velocity of flow and upon the discharge when the head is doubled? when the head is reduced one half? Is the same true for short pipes?

In such cases it is understood that factors not otherwise indicated remain constant. That is, in comparing the change in velocity caused by a change in head it is assumed that the diameter and the length of the pipe remain unchanged. The general laws governing in Exs. 9–11 should be determined from a study of Exs. 1–8, above, and of the formulas given on pages 286 and 287.

10. In long pipes, what is the effect upon the velocity of flow and upon the discharge when the diameter of the pipe is doubled? when the diameter is reduced one half? Is the same true for short pipes? Explain your answer.

11. In long pipes, what is the effect upon the velocity of flow and upon the discharge when the length of the pipe is doubled? when the length is reduced one half? Is the same true for short pipes? Explain your answer.

12. Under a head of 25 ft., what diameter of pipe will it be necessary to use in order to furnish 1000 gal. of water per hour through a length of 2500 ft.?

13. If the pressure in the water mains of a town is maintained at 18 lb./sq. in., how many gallons per hour will a $\frac{3}{4}$-inch pipe deliver at a distance of 25 ft. from the main?

14. A standpipe 40 ft. high supplies water through a 1-inch pipe to a camp 1 mi. away. If the standpipe is kept always full, how many persons requiring an average of 20 gal. of water per day can be accommodated in the camp?

15. A reservoir in which the water stands at a constant level of 20 ft. above the lower end of the supply pipe is to furnish a farm $\frac{1}{4}$ mi. away with 120 bbl. of water per day. Disregarding the friction at the entrance to the pipe, find the diameter of the pipe needed to deliver the specified amount.

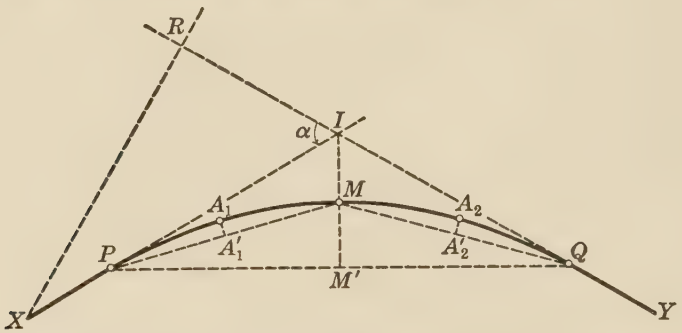
16. Which will deliver water the faster, a 1-inch pipe 40 ft. long under a head of 20 ft. or a $\frac{3}{4}$-inch pipe 100 ft. long under a pressure of 16 lb./sq. in.? What is the difference between the amounts discharged per hour from the two pipes?

17. Under a head of 120 ft., what diameter of pipe will it be necessary to use in order to deliver 2000 bbl. of water per day through a length of 5000 ft.?

18. Under a head of 64 ft., will a $1\frac{1}{2}$-inch pipe 3600 ft. long supply an excess or a shortage of water to a camp accommodating 800 persons requiring an average of 20 gal. of water per head per day? How much is the daily excess or shortage?

19. If a 2-inch pipe had been used in Ex. 18, how many more persons could have been accommodated in the camp? What size of pipe will be needed to accommodate 1200 persons?

Staking out a Circular Curve. In the construction of highways and ditches it is frequently necessary to lay out a circular curve of a given length connecting two straight lines. The following discussion sets forth a simple method of staking out such a curve, within practical limits of accuracy, without requiring the use of any surveying instrument except a tape.



Briefly stated, the method consists in locating points on the curve by measuring successive offsets from the midpoints of a number of chords of the arc. The following table, which is used in connection with the method, shows in the last column that the length of the first offset $M'M$ (the middle ordinate of the complete curve PMQ) is from 43 % to 50 % of the distance from the midpoint M' of the chord PQ to the intersection I of the two straight-line segments which are connected by the curve.

Length of IX in Feet	Length of XR in Feet	Length of IR in Feet	Size of Angle α	Per Cent of Curve Length for IP or IQ	Per Cent of $M'I$ for $M'M$
100	0–50.0	100–86.6	0°–30°	50%	50%
100	51.5–70.7	85.7–70.7	31°–45°	55%	49%
100	71.9–86.6	69.5–50.0	46°–60°	55%	48%
100	87.5–96.6	48.5–25.9	61°–75°	60%	47%
100	97.0–100	24.2–0	76°–90°	$66\frac{2}{3}$%	43%

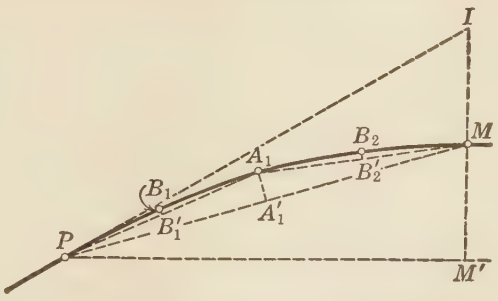
The uses of the other columns in the table and the detailed procedure in staking out the curve are given on the following page.

Locating the Points. The length of the curve is first determined from the practical conditions which affect the situation, but which we do not need to discuss here.

The next step is to locate the position of the intersection I (see the figure on page 292) of the straight-line segments which the curve is to connect. The position of X is then located with the tape, the length of IX being arbitrarily taken as 100 ft. for convenience. The line XR is then laid off perpendicular to YI produced, and XR and IR are measured roughly. From these measurements the size of angle α, which is equal to the central angle of the curve, is determined from the table.

From this value of α the positions of the end points P and Q are located by laying off IP and IQ equal to the per cent of the curve length indicated in the table. The midpoint M' of the chord PQ is then located, and the midpoint M of the arc is found by laying off on $M'I$ a distance $M'M$ equal to the per cent of $M'I$ indicated in the table.

The midpoints A'_1 and A'_2 of the chords PM and MQ are then located, and the points A_1 and A_2, the *quarter points* of the curve, are determined by taking the offsets (perpendiculars) A'_1A_1 and A'_2A_2 equal to $\frac{1}{4}M'M$. As shown in the figure at the right, which gives the portion PM of the curve enlarged, the points B_1 and B_2 of the curve are located from the midpoints of the respective subchords PA_1 and A_1M by taking the offsets B'_1B_1 and B'_2B_2 equal to $\frac{1}{4}$ A'_1A_1. This process may be continued until sufficient points have been located.



That is, the method depends upon taking the middle ordinates of any two arcs as varying as the squares of the lengths of the arcs, which is sufficiently accurate for certain types of work.

Exercises. Circular Curves

1. In the figure on page 292, what length should IR have for a curve that is to have a central angle (α) of 40° and a length of 30 ft.? What radius will the curve have? What length should be used in locating the end points P and Q? Determine the offsets MM', A'_1A_1, and B'_1B_1 needed in locating seven interior points and plot the curve to a convenient scale.

2. In laying out a circular curve which is to have a radius of 110 ft., the length of IR is found to be 50 ft. What is the approximate length of the curve? What length should be used in locating the end points P and Q? Determine the three offsets needed in order to locate seven interior points of the curve and plot the curve.

3. In laying out a curve which is to have a radius of 60 ft. and a length of 40 ft., what length should XR have? What length should be used for IP and IQ? Determine the four offsets $M'M$, A'_1A_1, B'_1B_1, and C'_1C_1 needed in order to locate 15 interior points of the curve and plot the curve.

4. In each of Exs. 1–3 determine by trigonometric methods the error, to the nearest 0.1 ft., in locating the end points of the curve by the offset method.

5. In each of Exs. 1–3 determine by trigonometric methods the location of the midpoint and the quarter points of a mathematically-exact curve. Determine whether the corresponding points located by the offset method lie inside or outside the true circular curve. Which do these errors tend to make the curve, — flatter or sharper than the true curve?

In order to decrease the tendency of curves to reduce the speed of travel or of water flow, it is considered desirable in railroad, highway, and ditch practice to spiral the ends of the curve; that is, to flatten the ends so that the curve blends gradually into the tangents. The answers to the last problem given above should reveal whether or not in cases where the offset method can be used a curve staked out in this way tends to fulfill this requirement of good engineering practice.

Exercises. Test on Chapter XIII

1. For a beam supported at one end and loaded at the other, what will be the effect upon the safe load in reducing the depth one half? in increasing the depth one half?

2. In Ex. 1, what effect will reducing the breadth of the beam one half have upon the safe load? What will be the effect if the breadth is increased one half?

3. A 30-foot length of basement wall 1 ft. thick and $7\frac{1}{2}$ ft. high is to be built with a footing course 2 ft. wide and 1 ft. deep. If a 1–3–5 mixture of concrete is used for the footing course, and a 1–2–4 mixture for the walls, how much cement, sand, and gravel are needed for the entire job?

4. A concrete retaining wall 50 ft. long is to be built with a foundation 5 ft. wide and 3 ft. deep of 1–3–5 mixture, and the main wall is to be made 12 ft. high, 3 ft. thick at the base, and 2 ft. thick on top, with a 1–2–4 mixture. How much cement, sand, and gravel are needed for the entire job?

5. In planning a farm water system it was found that it would be necessary to have a head of 25 ft. in order to supply the desired quantity through 625 ft. of 1-inch pipe. If a $1\frac{1}{4}$-inch pipe was used, because smaller pipe could not be obtained, what head would then be necessary?

6. Under a head of 40 ft., find the diameter of a 2500-foot length of pipe needed to supply a camp accommodating 1000 persons requiring 30 gal. of water per person per day. For the size of pipe thus determined, find the effect upon the discharge of reducing the head one half; of increasing it one fourth.

7. A circular curve with a radius of 200 ft. and a length of 150 ft. was laid out by the method described on pages 292 and 293. How far were the midpoint, quarter points, and eighth points from the corresponding points of a mathematically exact curve? State whether the points were inside or outside the true curve. Was the curve flattened at the ends?

Exercises. General Review

1. The horse power P of one type of motor-truck engine is given by the formula $P = 0.4\,d^2n$, where d is the diameter of each cylinder in inches and n is the number of cylinders. Find the value of P when $d = 4$ and $n = 4$.

2. The formula for the amount A of a sum invested at simple interest is $A = P(1 + it)$, where P is the principal in dollars, i is the rate of interest expressed in hundredths, and t is the time in years. Draw a graph of the formula for a principal of \$200 at 6%, and from it estimate the number of years in which the principal will double itself.

3. If $ax - by = 20$ is satisfied when $x = 2$ and $y = 7$, and also when $x = -2\frac{2}{3}$ and $y = 14$, what are the values of a and b?

4. The treads of the steps on a stairway are 10 in. wide, and each step rises 8 in. above the next one below it. Find the angle at which the stairway rises.

5. A box, open at the top, is made by cutting a 1-inch square from each corner of a square piece of tin and folding up the flaps. If the box thus made contains 36 cu. in., what was the length of side of the original piece of tin?

6. Prove that $\log a^n = n \log a$.

7. After writing 1.06^6 as $(1 + 0.06)^6$, expand by the Binomial Formula and find the value of 1.06^6 to the nearest 0.0001.

8. Find to the nearest 0.01 in. the length of side of the largest square that can be inscribed in a circle $5\frac{1}{2}$ in. in diameter; in one 6.84 in. in diameter.

9. If a man should pay off a debt of \$5040 in monthly payments by paying \$25 the first month and increasing this payment by \$2 each succeeding month, how long, disregarding interest, would it take him to pay off the debt?

10. Find the amount at the end of 12 yr. of \$2250 invested at 5%, compounded semiannually.

TABLES

1. Powers and Roots

Table of Square Roots, 1.00–99.9. The table on pages 298 and 299 gives, to the nearest 0.001, the square roots of numbers from 1.00 to 9.99, and the table on pages 300 and 301 gives the square roots of numbers from 10.0 to 99.9.

In finding the square root of 2.78, for example, we first look in the left-hand column on page 298 for 2.7. Then in the same line, in the column headed 8, we find 1.667; that is, $\sqrt{2.78} = 1.667$, to the nearest 0.001.

Since $\sqrt{100} = 10$ and $\sqrt{0.01} = 0.1$, we can extend the table to cases like finding the value of $\sqrt{278}$ and $\sqrt{0.0278}$.

For example, we have $\sqrt{278} = \sqrt{100 \times 2.78} = 10 \times 1.667 = 16.67$, and $\sqrt{0.0278} = \sqrt{0.01 \times 2.78} = 0.1 \times 1.667 = 0.1667$. In practice we simply look up $\sqrt{2.78}$ and then put the decimal point in its proper place while copying the result from the table.

The columns 1 to 9 at the right of each page are for use in cases where the given number has a fourth figure.

For example, as above, $\sqrt{2.78} = 1.667$. If we wish to find $\sqrt{2.782}$, we look in column 2 at the right on line 2.7 and find 1, which means that 0.001 is to be added to 1.667; that is,

$$\sqrt{2.782} = 1.667 + 0.001 = 1.668.$$

Table of Powers and Roots, 1–100. The table on page 302 gives in a convenient and compact form the squares, cubes, square roots, and cube roots of the integers from 1 to 100. The use of this table is practically self-evident.

For example, from the column headed "Square Roots" we easily see that $\sqrt{33} = 5.745$, while from the column headed "Cube Roots" we see that $\sqrt[3]{74} = 4.198$, each to the nearest 0.001.

In the column headed "Cubes" we also see that $4913 = 17^3$; that is, $\sqrt[3]{4913} = 17$. In finding square roots, the larger tables on pages 298–301 are generally more convenient.

SQUARE ROOTS. 1.00–5.49

N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
1.0	1.000	1.005	1.010	1.015	1.020	1.025	1.030	1.034	1.039	1.044	0 1 1	2 2 3	3 4 4
1.1	1.049	1.054	1.058	1.063	1.068	1.072	1.077	1.082	1.086	1.091	0 1 1	2 2 3	3 4 4
1.2	1.095	1.100	1.105	1.109	1.114	1.118	1.122	1.127	1.131	1.136	0 1 1	2 2 3	3 4 4
1.3	1.140	1.145	1.149	1.153	1.158	1.162	1.166	1.170	1.175	1.179	0 1 1	2 2 3	3 3 4
1.4	1.183	1.187	1.192	1.196	1.200	1.204	1.208	1.212	1.217	1.221	0 1 1	2 2 2	3 3 4
1.5	1.225	1.229	1.233	1.237	1.241	1.245	1.249	1.253	1.257	1.261	0 1 1	2 2 2	3 3 4
1.6	1.265	1.269	1.273	1.277	1.281	1.285	1.288	1.292	1.296	1.300	0 1 1	2 2 2	3 3 4
1.7	1.304	1.308	1.311	1.315	1.319	1.323	1.327	1.330	1.334	1.338	0 1 1	2 2 2	3 3 3
1.8	1.342	1.345	1.349	1.353	1.356	1.360	1.364	1.367	1.371	1.375	0 1 1	1 2 2	3 3 3
1.9	1.378	1.382	1.386	1.389	1.393	1.396	1.400	1.404	1.407	1.411	0 1 1	1 2 2	3 3 3
2.0	1.414	1.418	1.421	1.425	1.428	1.432	1.435	1.439	1.442	1.446	0 1 1	1 2 2	2 3 3
2.1	1.449	1.453	1.456	1.459	1.463	1.466	1.470	1.473	1.476	1.480	0 1 1	1 2 2	2 3 3
2.2	1.483	1.487	1.490	1.493	1.497	1.500	1.503	1.507	1.510	1.513	0 1 1	1 2 2	2 3 3
2.3	1.517	1.520	1.523	1.526	1.530	1.533	1.536	1.539	1.543	1.546	0 1 1	1 2 2	2 3 3
2.4	1.549	1.552	1.556	1.559	1.562	1.565	1.568	1.572	1.575	1.578	0 1 1	1 2 2	2 3 3
2.5	1.581	1.584	1.587	1.591	1.594	1.597	1.600	1.603	1.606	1.609	0 1 1	1 2 2	2 3 3
2.6	1.612	1.616	1.619	1.622	1.625	1.628	1.631	1.634	1.637	1.640	0 1 1	1 2 2	2 2 3
2.7	1.643	1.646	1.649	1.652	1.655	1.658	1.661	1.664	1.667	1.670	0 1 1	1 2 2	2 2 3
2.8	1.673	1.676	1.679	1.682	1.685	1.688	1.691	1.694	1.697	1.700	0 1 1	1 1 2	2 2 3
2.9	1.703	1.706	1.709	1.712	1.715	1.718	1.720	1.723	1.726	1.729	0 1 1	1 1 2	2 2 3
3.0	1.732	1.735	1.738	1.741	1.744	1.746	1.749	1.752	1.755	1.758	0 1 1	1 1 2	2 2 3
3.1	1.761	1.764	1.766	1.769	1.772	1.775	1.778	1.780	1.783	1.786	0 1 1	1 1 2	2 2 3
3.2	1.789	1.792	1.794	1.797	1.800	1.803	1.806	1.808	1.811	1.814	0 1 1	1 1 2	2 2 2
3.3	1.817	1.819	1.822	1.825	1.828	1.830	1.833	1.836	1.838	1.841	0 1 1	1 1 2	2 2 2
3.4	1.844	1.847	1.849	1.852	1.855	1.857	1.860	1.863	1.865	1.868	0 1 1	1 1 2	2 2 2
3.5	1.871	1.873	1.876	1.879	1.881	1.884	1.887	1.889	1.892	1.895	0 1 1	1 1 2	2 2 2
3.6	1.897	1.900	1.903	1.905	1.908	1.910	1.913	1.916	1.918	1.921	0 1 1	1 1 2	2 2 2
3.7	1.924	1.926	1.929	1.931	1.934	1.936	1.939	1.942	1.944	1.947	0 1 1	1 1 2	2 2 2
3.8	1.949	1.952	1.954	1.957	1.960	1.962	1.965	1.967	1.970	1.972	0 1 1	1 1 2	2 2 2
3.9	1.975	1.977	1.980	1.982	1.985	1.987	1.990	1.992	1.995	1.997	0 1 1	1 1 2	2 2 2
4.0	2.000	2.002	2.005	2.007	2.010	2.012	2.015	2.017	2.020	2.022	0 0 1	1 1 1	2 2 2
4.1	2.025	2.027	2.030	2.032	2.035	2.037	2.040	2.042	2.045	2.047	0 0 1	1 1 1	2 2 2
4.2	2.049	2.052	2.054	2.057	2.059	2.062	2.064	2.066	2.069	2.071	0 0 1	1 1 1	2 2 2
4.3	2.074	2.076	2.078	2.081	2.083	2.086	2.088	2.090	2.093	2.095	0 0 1	1 1 1	2 2 2
4.4	2.098	2.100	2.102	2.105	2.107	2.110	2.112	2.114	2.117	2.119	0 0 1	1 1 1	2 2 2
4.5	2.121	2.124	2.126	2.128	2.131	2.133	2.135	2.138	2.140	2.142	0 0 1	1 1 1	2 2 2
4.6	2.145	2.147	2.149	2.152	2.154	2.156	2.159	2.161	2.163	2.166	0 0 1	1 1 1	2 2 2
4.7	2.168	2.170	2.173	2.175	2.177	2.179	2.182	2.184	2.186	2.189	0 0 1	1 1 1	2 2 2
4.8	2.191	2.193	2.195	2.198	2.200	2.202	2.205	2.207	2.209	2.211	0 0 1	1 1 1	2 2 2
4.9	2.214	2.216	2.218	2.220	2.223	2.225	2.227	2.229	2.232	2.234	0 0 1	1 1 1	2 2 2
5.0	2.236	2.238	2.241	2.243	2.245	2.247	2.249	2.252	2.254	2.256	0 0 1	1 1 1	2 2 2
5.1	2.258	2.261	2.263	2.265	2.267	2.269	2.272	2.274	2.276	2.278	0 0 1	1 1 1	2 2 2
5.2	2.280	2.283	2.285	2.287	2.289	2.291	2.293	2.296	2.298	2.300	0 0 1	1 1 1	2 2 2
5.3	2.302	2.304	2.307	2.309	2.311	2.313	2.315	2.317	2.319	2.322	0 0 1	1 1 1	2 2 2
5.4	2.324	2.326	2.328	2.330	2.332	2.335	2.337	2.339	2.341	2.343	0 0 1	1 1 1	1 2 2

The figures in the difference columns are to be added.

SQUARE ROOTS. 5.50–9.99

N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
5.5	2.345	2.347	2.349	2.352	2.354	2.356	2.358	2.360	2.362	2.364	0 0 1	1 1 1	1 2 2
5.6	2.366	2.369	2.371	2.373	2.375	2.377	2.379	2.381	2.383	2.385	0 0 1	1 1 1	1 2 2
5.7	2.387	2.390	2.392	2.394	2.396	2.398	2.400	2.402	2.404	2.406	0 0 1	1 1 1	1 2 2
5.8	2.408	2.410	2.412	2.415	2.417	2.419	2.421	2.423	2.425	2.427	0 0 1	1 1 1	1 2 2
5.9	2.429	2.431	2.433	2.435	2.437	2.439	2.441	2.443	2.445	2.447	0 0 1	1 1 1	1 2 2
6.0	2.449	2.452	2.454	2.456	2.458	2.460	2.462	2.464	2.466	2.468	0 0 1	1 1 1	1 2 2
6.1	2.470	2.472	2.474	2.476	2.478	2.480	2.482	2.484	2.486	2.488	0 0 1	1 1 1	1 2 2
6.2	2.490	2.492	2.494	2.496	2.498	2.500	2.502	2.504	2.506	2.508	0 0 1	1 1 1	1 2 2
6.3	2.510	2.512	2.514	2.516	2.518	2.520	2.522	2.524	2.526	2.528	0 0 1	1 1 1	1 2 2
6.4	2.530	2.532	2.534	2.536	2.538	2.540	2.542	2.544	2.546	2.548	0 0 1	1 1 1	1 2 2
6.5	2.550	2.551	2.553	2.555	2.557	2.559	2.561	2.563	2.565	2.567	0 0 1	1 1 1	1 2 2
6.6	2.569	2.571	2.573	2.575	2.577	2.579	2.581	2.583	2.585	2.587	0 0 1	1 1 1	1 2 2
6.7	2.588	2.590	2.592	2.594	2.596	2.598	2.600	2.602	2.604	2.606	0 0 1	1 1 1	1 2 2
6.8	2.608	2.610	2.612	2.613	2.615	2.617	2.619	2.621	2.623	2.625	0 0 1	1 1 1	1 2 2
6.9	2.627	2.629	2.631	2.632	2.634	2.636	2.638	2.640	2.642	2.644	0 0 1	1 1 1	1 2 2
7.0	2.646	2.648	2.650	2.651	2.653	2.655	2.657	2.659	2.661	2.663	0 0 1	1 1 1	1 2 2
7.1	2.665	2.666	2.668	2.670	2.672	2.674	2.676	2.678	2.680	2.681	0 0 1	1 1 1	1 1 2
7.2	2.683	2.685	2.687	2.689	2.691	2.693	2.694	2.696	2.698	2.700	0 0 1	1 1 1	1 1 2
7.3	2.702	2.704	2.706	2.707	2.709	2.711	2.713	2.715	2.717	2.718	0 0 1	1 1 1	1 1 2
7.4	2.720	2.722	2.724	2.726	2.728	2.729	2.731	2.733	2.735	2.737	0 0 1	1 1 1	1 1 2
7.5	2.739	2.740	2.742	2.744	2.746	2.748	2.750	2.751	2.753	2.755	0 0 1	1 1 1	1 1 2
7.6	2.757	2.759	2.760	2.762	2.764	2.766	2.768	2.769	2.771	2.773	0 0 1	1 1 1	1 1 2
7.7	2.775	2.777	2.778	2.780	2.782	2.784	2.786	2.787	2.789	2.791	0 0 1	1 1 1	1 1 2
7.8	2.793	2.795	2.796	2.798	2.800	2.802	2.804	2.805	2.807	2.809	0 0 1	1 1 1	1 1 2
7.9	2.811	2.812	2.814	2.816	2.818	2.820	2.821	2.823	2.825	2.827	0 0 1	1 1 1	1 1 2
8.0	2.828	2.830	2.832	2.834	2.835	2.837	2.839	2.841	2.843	2.844	0 0 1	1 1 1	1 1 2
8.1	2.846	2.848	2.850	2.851	2.853	2.855	2.857	2.858	2.860	2.862	0 0 1	1 1 1	1 1 2
8.2	2.864	2.865	2.867	2.869	2.871	2.872	2.874	2.876	2.877	2.879	0 0 1	1 1 1	1 1 2
8.3	2.881	2.883	2.884	2.886	2.888	2.890	2.891	2.893	2.895	2.897	0 0 1	1 1 1	1 1 2
8.4	2.898	2.900	2.902	2.903	2.905	2.907	2.909	2.910	2.912	2.914	0 0 1	1 1 1	1 1 2
8.5	2.915	2.917	2.919	2.921	2.922	2.924	2.926	2.927	2.929	2.931	0 0 1	1 1 1	1 1 2
8.6	2.933	2.934	2.936	2.938	2.939	2.941	2.943	2.944	2.946	2.948	0 0 1	1 1 1	1 1 2
8.7	2.950	2.951	2.953	2.955	2.956	2.958	2.960	2.961	2.963	2.965	0 0 1	1 1 1	1 1 2
8.8	2.966	2.968	2.970	2.972	2.973	2.975	2.977	2.978	2.980	2.982	0 0 1	1 1 1	1 1 2
8.9	2.983	2.985	2.987	2.988	2.990	2.992	2.993	2.995	2.997	2.998	0 0 1	1 1 1	1 1 2
9.0	3.000	3.002	3.003	3.005	3.007	3.008	3.010	3.012	3.013	3.015	0 0 0	1 1 1	1 1 1
9.1	3.017	3.018	3.020	3.022	3.023	3.025	3.027	3.028	3.030	3.032	0 0 0	1 1 1	1 1 1
9.2	3.033	3.035	3.036	3.038	3.040	3.041	3.043	3.045	3.046	3.048	0 0 0	1 1 1	1 1 1
9.3	3.050	3.051	3.053	3.055	3.056	3.058	3.059	3.061	3.063	3.064	0 0 0	1 1 1	1 1 1
9.4	3.066	3.068	3.069	3.071	3.072	3.074	3.076	3.077	3.079	3.081	0 0 0	1 1 1	1 1 1
9.5	3.082	3.084	3.085	3.087	3.089	3.090	3.092	3.094	3.095	3.097	0 0 0	1 1 1	1 1 1
9.6	3.098	3.100	3.102	3.103	3.105	3.106	3.108	3.110	3.111	3.113	0 0 0	1 1 1	1 1 1
9.7	3.114	3.116	3.118	3.119	3.121	3.122	3.124	3.126	3.127	3.129	0 0 0	1 1 1	1 1 1
9.8	3.130	3.132	3.134	3.135	3.137	3.138	3.140	3.142	3.143	3.145	0 0 0	1 1 1	1 1 1
9.9	3.146	3.148	3.150	3.151	3.153	3.154	3.156	3.158	3.159	3.161	0 0 0	1 1 1	1 1 1

The figures in the difference columns are to be added.

SQUARE ROOTS. 10.0–54.9

N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
10	3.162	3.178	3.194	3.209	3.225	3.240	3.256	3.271	3.286	3.302	2 3 5	6 8 9	11 12 14
11	3.317	3.332	3.347	3.362	3.376	3.391	3.406	3.421	3.435	3.450	1 3 4	6 7 9	10 12 13
12	3.464	3.479	3.493	3.507	3.521	3.536	3.550	3.564	3.578	3.592	1 3 4	6 7 8	10 11 13
13	3.606	3.619	3.633	3.647	3.661	3.674	3.688	3.701	3.715	3.728	1 3 4	5 7 8	10 11 12
14	3.742	3.755	3.768	3.782	3.795	3.808	3.821	3.834	3.847	3.860	1 3 4	5 7 8	9 11 12
15	3.873	3.886	3.899	3.912	3.924	3.937	3.950	3.962	3.975	3.987	1 3 4	5 6 8	9 10 11
16	4.000	4.012	4.025	4.037	4.050	4.062	4.074	4.087	4.099	4.111	1 2 4	5 6 7	9 10 11
17	4.123	4.135	4.147	4.159	4.171	4.183	4.195	4.207	4.219	4.231	1 2 4	5 6 7	8 10 11
18	4.243	4.254	4.266	4.278	4.290	4.301	4.313	4.324	4.336	4.347	1 2 3	5 6 7	8 9 10
19	4.359	4.370	4.382	4.393	4.405	4.416	4.427	4.438	4.450	4.461	1 2 3	5 6 7	8 9 10
20	4.472	4.483	4.494	4.506	4.517	4.528	4.539	4.550	4.561	4.572	1 2 3	4 6 7	8 9 10
21	4.583	4.593	4.604	4.615	4.626	4.637	4.648	4.658	4.669	4.680	1 2 3	4 5 6	8 9 10
22	4.690	4.701	4.712	4.722	4.733	4.743	4.754	4.764	4.775	4.785	1 2 3	4 5 6	7 8 9
23	4.796	4.806	4.817	4.827	4.837	4.848	4.858	4.868	4.879	4.889	1 2 3	4 5 6	7 8 9
24	4.899	4.909	4.919	4.930	4.940	4.950	4.960	4.970	4.980	4.990	1 2 3	4 5 6	7 8 9
25	5.000	5.010	5.020	5.030	5.040	5.050	5.060	5.070	5.079	5.089	1 2 3	4 5 6	7 8 9
26	5.099	5.109	5.119	5.128	5.138	5.148	5.158	5.167	5.177	5.187	1 2 3	4 5 6	7 8 9
27	5.196	5.206	5.215	5.225	5.235	5.244	5.254	5.263	5.273	5.282	1 2 3	4 5 6	7 8 9
28	5.292	5.301	5.310	5.320	5.329	5.339	5.348	5.357	5.367	5.376	1 2 3	4 5 6	7 7 8
29	5.385	5.394	5.404	5.413	5.422	5.431	5.441	5.450	5.459	5.468	1 2 3	4 5 5	6 7 8
30	5.477	5.486	5.495	5.505	5.514	5.523	5.532	5.541	5.550	5.559	1 2 3	4 4 5	6 7 8
31	5.568	5.577	5.586	5.595	5.604	5.612	5.621	5.630	5.639	5.648	1 2 3	3 4 5	6 7 8
32	5.657	5.666	5.675	5.683	5.692	5.701	5.710	5.718	5.727	5.736	1 2 3	3 4 5	6 7 8
33	5.745	5.753	5.762	5.771	5.779	5.788	5.797	5.805	5.814	5.822	1 2 3	3 4 5	6 7 8
34	5.831	5.840	5.848	5.857	5.865	5.874	5.882	5.891	5.899	5.908	1 2 3	3 4 5	6 7 8
35	5.916	5.925	5.933	5.941	5.950	5.958	5.967	5.975	5.983	5.992	1 2 2	3 4 5	6 7 8
36	6.000	6.008	6.017	6.025	6.033	6.042	6.050	6.058	6.066	6.075	1 2 2	3 4 5	6 7 7
37	6.083	6.091	6.099	6.107	6.116	6.124	6.132	6.140	6.148	6.156	1 2 2	3 4 5	6 7 7
38	6.164	6.173	6.181	6.189	6.197	6.205	6.213	6.221	6.229	6.237	1 2 2	3 4 5	6 6 7
39	6.245	6.253	6.261	6.269	6.277	6.285	6.293	6.301	6.309	6.317	1 2 2	3 4 5	6 6 7
40	6.325	6.332	6.340	6.348	6.356	6.364	6.372	6.380	6.387	6.395	1 2 2	3 4 5	6 6 7
41	6.403	6.411	6.419	6.427	6.434	6.442	6.450	6.458	6.465	6.473	1 2 2	3 4 5	5 6 7
42	6.481	6.488	6.496	6.504	6.512	6.519	6.527	6.535	6.542	6.550	1 2 2	3 4 5	5 6 7
43	6.557	6.565	6.573	6.580	6.588	6.595	6.603	6.611	6.618	6.626	1 2 2	3 4 5	5 6 7
44	6.633	6.641	6.648	6.656	6.663	6.671	6.678	6.686	6.693	6.701	1 2 2	3 4 5	5 6 7
45	6.708	6.716	6.723	6.731	6.738	6.745	6.753	6.760	6.768	6.775	1 1 2	3 4 4	5 6 7
46	6.782	6.790	6.797	6.804	6.812	6.819	6.826	6.834	6.841	6.848	1 1 2	3 4 4	5 6 7
47	6.856	6.863	6.870	6.877	6.885	6.892	6.899	6.907	6.914	6.921	1 1 2	3 4 4	5 6 7
48	6.928	6.935	6.943	6.950	6.957	6.964	6.971	6.979	6.986	6.993	1 1 2	3 4 4	5 6 6
49	7.000	7.007	7.014	7.021	7.029	7.036	7.043	7.050	7.057	7.064	1 1 2	3 4 4	5 6 6
50	7.071	7.078	7.085	7.092	7.099	7.106	7.113	7.120	7.127	7.134	1 1 2	3 4 4	5 6 6
51	7.141	7.148	7.155	7.162	7.169	7.176	7.183	7.190	7.197	7.204	1 1 2	3 4 4	5 6 6
52	7.211	7.218	7.225	7.232	7.239	7.246	7.253	7.259	7.266	7.273	1 1 2	3 3 4	5 6 6
53	7.280	7.287	7.294	7.301	7.308	7.314	7.321	7.328	7.335	7.342	1 1 2	3 3 4	5 5 6
54	7.348	7.355	7.362	7.369	7.376	7.382	7.389	7.396	7.403	7.409	1 1 2	3 3 4	5 5 6

The figures in the difference columns are to be added.

SQUARE ROOTS. 55.0–99.9

N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
55	7.416	7.423	7.430	7.436	7.443	7.450	7.457	7.463	7.470	7.477	1 1 2	3 3 4	5 5 6
56	7.483	7.490	7.497	7.503	7.510	7.517	7.523	7.530	7.537	7.543	1 1 2	3 3 4	5 5 6
57	7.550	7.556	7.563	7.570	7.576	7.583	7.589	7.596	7.603	7.609	1 1 2	3 3 4	5 5 6
58	7.616	7.622	7.629	7.635	7.642	7.649	7.655	7.662	7.668	7.675	1 1 2	3 3 4	5 5 6
59	7.681	7.688	7.694	7.701	7.707	7.714	7.720	7.727	7.733	7.740	1 1 2	3 3 4	4 5 6
60	7.746	7.752	7.759	7.765	7.772	7.778	7.785	7.791	7.797	7.804	1 1 2	3 3 4	4 5 6
61	7.810	7.817	7.823	7.829	7.836	7.842	7.849	7.855	7.861	7.868	1 1 2	3 3 4	4 5 6
62	7.874	7.880	7.887	7.893	7.899	7.906	7.912	7.918	7.925	7.931	1 1 2	3 3 4	4 5 6
63	7.937	7.944	7.950	7.956	7.962	7.969	7.975	7.981	7.987	7.994	1 1 2	3 3 4	4 5 6
64	8.000	8.006	8.012	8.019	8.025	8.031	8.037	8.044	8.050	8.056	1 1 2	2 3 4	4 5 6
65	8.062	8.068	8.075	8.081	8.087	8.093	8.099	8.106	8.112	8.118	1 1 2	2 3 4	4 5 5
66	8.124	8.130	8.136	8.142	8.149	8.155	8.161	8.167	8.173	8.179	1 1 2	2 3 4	4 5 5
67	8.185	8.191	8.198	8.204	8.210	8.216	8.222	8.228	8.234	8.240	1 1 2	2 3 4	4 5 5
68	8.246	8.252	8.258	8.264	8.270	8.276	8.283	8.289	8.295	8.301	1 1 2	2 3 4	4 5 5
69	8.307	8.313	8.319	8.325	8.331	8.337	8.343	8.349	8.355	8.361	1 1 2	2 3 4	4 5 5
70	8.367	8.373	8.379	8.385	8.390	8.396	8.402	8.408	8.414	8.420	1 1 2	2 3 4	4 5 5
71	8.426	8.432	8.438	8.444	8.450	8.456	8.462	8.468	8.473	8.479	1 1 2	2 3 4	4 5 5
72	8.485	8.491	8.497	8.503	8.509	8.515	8.521	8.526	8.532	8.538	1 1 2	2 3 3	4 5 5
73	8.544	8.550	8.556	8.562	8.567	8.573	8.579	8.585	8.591	8.597	1 1 2	2 3 3	4 5 5
74	8.602	8.608	8.614	8.620	8.626	8.631	8.637	8.643	8.649	8.654	1 1 2	2 3 3	4 5 5
75	8.660	8.666	8.672	8.678	8.683	8.689	8.695	8.701	8.706	8.712	1 1 2	2 3 3	4 5 5
76	8.718	8.724	8.729	8.735	8.741	8.746	8.752	8.758	8.764	8.769	1 1 2	2 3 3	4 5 5
77	8.775	8.781	8.786	8.792	8.798	8.803	8.809	8.815	8.820	8.826	1 1 2	2 3 3	4 4 5
78	8.832	8.837	8.843	8.849	8.854	8.860	8.866	8.871	8.877	8.883	1 1 2	2 3 3	4 4 5
79	8.888	8.894	8.899	8.905	8.911	8.916	8.922	8.927	8.933	8.939	1 1 2	2 3 3	4 4 5
80	8.944	8.950	8.955	8.961	8.967	8.972	8.978	8.983	8.989	8.994	1 1 2	2 3 3	4 4 5
81	9.000	9.006	9.011	9.017	9.022	9.028	9.033	9.039	9.044	9.050	1 1 2	2 3 3	4 4 5
82	9.055	9.061	9.066	9.072	9.077	9.083	9.088	9.094	9.099	9.105	1 1 2	2 3 3	4 4 5
83	9.110	9.116	9.121	9.127	9.132	9.138	9.143	9.149	9.154	9.160	1 1 2	2 3 3	4 4 5
84	9.165	9.171	9.176	9.182	9.187	9.192	9.198	9.203	9.209	9.214	1 1 2	2 3 3	4 4 5
85	9.220	9.225	9.230	9.236	9.241	9.247	9.252	9.257	9.263	9.268	1 1 2	2 3 3	4 4 5
86	9.274	9.279	9.284	9.290	9.295	9.301	9.306	9.311	9.317	9.322	1 1 2	2 3 3	4 4 5
87	9.327	9.333	9.338	9.343	9.349	9.354	9.359	9.365	9.370	9.375	1 1 2	2 3 3	4 4 5
88	9.381	9.386	9.391	9.397	9.402	9.407	9.413	9.418	9.423	9.429	1 1 2	2 3 3	4 4 5
89	9.434	9.439	9.445	9.450	9.455	9.460	9.466	9.471	9.476	9.482	1 1 2	2 3 3	4 4 5
90	9.487	9.492	9.497	9.503	9.508	9.513	9.518	9.524	9.529	9.534	1 1 2	2 3 3	4 4 5
91	9.539	9.545	9.550	9.555	9.560	9.566	9.571	9.576	9.581	9.586	1 1 2	2 3 3	4 4 5
92	9.592	9.597	9.602	9.607	9.612	9.618	9.623	9.628	9.633	9.638	1 1 2	2 3 3	4 4 5
93	9.644	9.649	9.654	9.659	9.664	9.670	9.675	9.680	9.685	9.690	1 1 2	2 3 3	4 4 5
94	9.695	9.701	9.706	9.711	9.716	9.721	9.726	9.731	9.737	9.742	1 1 2	2 3 3	4 4 5
95	9.747	9.752	9.757	9.762	9.767	9.772	9.778	9.783	9.788	9.793	1 1 2	2 3 3	4 4 5
96	9.798	9.803	9.808	9.813	9.818	9.823	9.829	9.834	9.839	9.844	1 1 2	2 3 3	4 4 5
97	9.849	9.854	9.859	9.864	9.869	9.874	9.879	9.884	9.889	9.894	1 1 2	2 3 3	4 4 5
98	9.899	9.905	9.910	9.915	9.920	9.925	9.930	9.935	9.940	9.945	0 1 1	2 2 3	3 4 4
99	9.950	9.955	9.960	9.965	9.970	9.975	9.980	9.985	9.990	9.995	0 1 1	2 2 3	3 4 4

The figures in the difference columns are to be added.

POWERS AND ROOTS. 1-100

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2 601	132 651	7.141	3.708
2	4	8	1.414	1.260	52	2 704	140 608	7.211	3.733
3	9	27	1.732	1.442	53	2 809	148 877	7.280	3.756
4	16	64	2.000	1.587	54	2 916	157 464	7.348	3.780
5	25	125	2.236	1.710	55	3 025	166 375	7.416	3.803
6	36	216	2.449	1.817	56	3 136	175 616	7.483	3.826
7	49	343	2.646	1.913	57	3 249	185 193	7.550	3.849
8	64	512	2.828	2.000	58	3 364	195 112	7.616	3.871
9	81	729	3.000	2.080	59	3 481	205 379	7.681	3.893
10	100	1 000	3.162	2.154	60	3 600	216 000	7.746	3.915
11	121	1 331	3.317	2.224	61	3 721	226 981	7.810	3.936
12	144	1 728	3.464	2.289	62	3 844	238 328	7.874	3.958
13	169	2 197	3.606	2.351	63	3 969	250 047	7.937	3.979
14	196	2 744	3.742	2.410	64	4 096	262 144	8.000	4.000
15	225	3 375	3.873	2.466	65	4 225	274 625	8.062	4.021
16	256	4 096	4.000	2.520	66	4 356	287 496	8.124	4.041
17	289	4 913	4.123	2.571	67	4 489	300 763	8.185	4.062
18	324	5 832	4.243	2.621	68	4 624	314 432	8.246	4.082
19	361	6 859	4.359	2.668	69	4 761	328 509	8.307	4.102
20	400	8 000	4.472	2.714	70	4 900	343 000	8.367	4.121
21	441	9 261	4.583	2.759	71	5 041	357 911	8.426	4.141
22	484	10 648	4.690	2.802	72	5 184	373 248	8.485	4.160
23	529	12 167	4.796	2.844	73	5 329	389 017	8.544	4.179
24	576	13 824	4.899	2.884	74	5 476	405 224	8.602	4.198
25	625	15 625	5.000	2.924	75	5 625	421 875	8.660	4.217
26	676	17 576	5.099	2.962	76	5 776	438 976	8.718	4.236
27	729	19 683	5.196	3.000	77	5 929	456 533	8.775	4.254
28	784	21 952	5.292	3.037	78	6 084	474 552	8.832	4.273
29	841	24 389	5.385	3.072	79	6 241	493 039	8.888	4.291
30	900	27 000	5.477	3.107	80	6 400	512 000	8.944	4.309
31	961	29 791	5.568	3.141	81	6 561	531 441	9.000	4.327
32	1 024	32 768	5.657	3.175	82	6 724	551 368	9.055	4.344
33	1 089	35 937	5.745	3.208	83	6 889	571 787	9.110	4.362
34	1 156	39 304	5.831	3.240	84	7 056	592 704	9.165	4.380
35	1 225	42 875	5.916	3.271	85	7 225	614 125	9.220	4.397
36	1 296	46 656	6.000	3.302	86	7 396	636 056	9.274	4.414
37	1 369	50 653	6.083	3.332	87	7 569	658 503	9.327	4.431
38	1 444	54 872	6.164	3.362	88	7 744	681 472	9.381	4.448
39	1 521	59 319	6.245	3.391	89	7 921	704 969	9.434	4.465
40	1 600	64 000	6.325	3.420	90	8 100	729 000	9.487	4.481
41	1 681	68 921	6.403	3.448	91	8 281	753 571	9.539	4.498
42	1 764	74 088	6.481	3.476	92	8 464	778 688	9.592	4.514
43	1 849	79 507	6.557	3.503	93	8 649	804 357	9.644	4.531
44	1 936	85 184	6.633	3.530	94	8 836	830 584	9.695	4.547
45	2 025	91 125	6.708	3.557	95	9 025	857 375	9.747	4.563
46	2 116	97 336	6.782	3.583	96	9 216	884 736	9.798	4.579
47	2 209	103 823	6.856	3.609	97	9 409	912 673	9.849	4.595
48	2 304	110 592	6.928	3.634	98	9 604	941 192	9.899	4.610
49	2 401	117 649	7.000	3.659	99	9 801	970 299	9.950	4.626
50	2 500	125 000	7.071	3.684	100	10 000	1 000 000	10.000	4.642

For the square roots of 1.00–99.9, see pages 298–301.

2. Investment Tables

Compound Amount of $1. The table on pages 304 and 305 shows values to seven decimal places of the factor $(1+i)^n$ in the compound-interest formula $A = P(1+i)^n$ given on page 148.

For example, in finding the compound amount of $2500 for 10 yr. at 4 % compounded annually, we find from page 305 that $(1+i)^n = 1.4802443$ when $i = 4\,\%$ and $n = 10$, and hence $A = 2500 \times 1.4802443 = 3700.61075$; that is, to the nearest cent, $A = \$3700.61$.

For values of i and n not given in the table the six-place table of logarithms on pages 316–333 should be used.

Present Value of $1. The table on pages 306 and 307 shows values of the factor $\dfrac{1}{(1+i)^n}$ in the compound-interest formula $P = A\,\dfrac{1}{(1+i)^n}$ given on page 154.

For example, in finding the sum that must be invested at 5 %, compounded semiannually, in order to accumulate to $5000 in 10 yr., we take i as $2\frac{1}{2}\,\%$ and n as 20. From page 306 we find that the factor is 0.6102709 for these values of i and n, and hence we have $P = 5000 \times 0.6102709 = 3051.3545$, or $P = \$3051.35$.

Amount of $1 per Annum. The table on pages 308 and 309 shows values of the factor $\dfrac{(1+i)^n - 1}{i}$ in the annuity formula $A_{\overline{n}|} = R\,\dfrac{(1+i)^n - 1}{i}$ given on page 156.

For example, in finding the value at the end of 20 yr. of an annuity of $1200 a year at 3 %, compounded annually, we find from page 308 of the table that the factor is 26.8703745 when $i = 3\,\%$ and $n = 20$. We therefore have $A_{\overline{n}|} = 1200 \times 26.8703745 = 32{,}244.44940$, or $A_{\overline{n}|} = \$32{,}244.45$.

Present Value of $1 per Annum. The table on pages 310 and 311 shows values of the factor $\dfrac{1}{i}\left[1 - \dfrac{1}{(1+i)^n}\right]$ in the annuity formula $P_{\overline{n}|} = R \cdot \dfrac{1}{i}\left[1 - \dfrac{1}{(1+i)^n}\right]$ given on page 157.

For example, in finding the cash value of the above annuity, we find from page 310 that the factor is 14.8774749 for $i = 3\,\%$ and $n = 20$, and hence $P_{\overline{n}|} = 1200 \times 14.8774749 = 17{,}852.96988$, or $P_{\overline{n}|} = \$17{,}852.97$.

These tables are reproduced from *The Mathematical Theory of Investment*, Revised Edition, by Professor Ernest B. Skinner (Ginn and Company).

COMPOUND AMOUNT (A) OF $1

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	1.012 5000	1.015 0000	1.020 0000	1.025 0000	1.030 0000	1
2	1.025 1563	1.030 2250	1.040 4000	1.050 6250	1.060 9000	2
3	1.037 9707	1.045 6784	1.061 2080	1.076 8906	1.092 7270	3
4	1.050 9453	1.061 3636	1.082 4322	1.103 8129	1.125 5088	4
5	1.064 0822	1.077 2840	1.104 0808	1.131 4082	1.159 2741	5
6	1.077 3832	1.093 4433	1.126 1624	1.159 6934	1.194 0523	6
7	1.090 8505	1.109 8449	1.148 6857	1.188 6858	1.229 8739	7
8	1.104 4861	1.126 4926	1.171 6594	1.218 4029	1.266 7701	8
9	1.118 2922	1.143 3900	1.195 0926	1.248 8630	1.304 7732	9
10	1.132 2708	1.160 5408	1.218 9944	1.280 0845	1.343 9164	10
11	1.146 4242	1.177 9489	1.243 3743	1.312 0867	1.384 2339	11
12	1.160 7545	1.195 6182	1.268 2418	1.344 8888	1.425 7609	12
13	1.175 2639	1.213 5524	1.293 6066	1.378 5110	1.468 5337	13
14	1.189 9547	1.231 7557	1.319 4788	1.412 9738	1.512 5897	14
15	1.204 8292	1.250 2321	1.345 8683	1.448 2982	1.557 9674	15
16	1.219 8895	1.268 9855	1.372 7857	1.484 5056	1.604 7064	16
17	1.235 1382	1.288 0203	1.400 2414	1.521 6183	1.652 8476	17
18	1.250 5774	1.307 3406	1.428 2463	1.559 6587	1.702 4331	18
19	1.266 2096	1.326 9507	1.456 8112	1.598 6502	1.753 5061	19
20	1.282 0372	1.346 8550	1.485 9474	1.638 6164	1.806 1112	20
21	1.298 0627	1.367 0578	1.515 6663	1.679 5819	1.860 2946	21
22	1.314 2885	1.387 5637	1.545 9797	1.721 5714	1.916 1034	22
23	1.330 7171	1.408 3771	1.576 8993	1.764 6107	1.973 5865	23
24	1.347 3511	1.429 5028	1.608 4373	1.808 7260	2.032 7941	24
25	1.364 1929	1.450 9454	1.640 6060	1.853 9441	2.093 7780	25
26	1.381 2454	1.472 7095	1.673 4181	1.900 2927	2.156 5913	26
27	1.398 5109	1.494 8002	1.706 8865	1.947 8000	2.221 2890	27
28	1.415 9923	1.517 2222	1.741 0242	1.996 4950	2.287 9277	28
29	1.433 6922	1.539 9805	1.775 8447	2.046 4074	2.356 5655	29
30	1.451 6134	1.563 0802	1.811 3616	2.097 5676	2.427 2625	30
31	1.469 7585	1.586 5264	1.847 5888	2.150 0068	2.500 0803	31
32	1.488 1305	1.610 3243	1.884 5406	2.203 7569	2.575 0828	32
33	1.506 7321	1.634 4792	1.922 2314	2.258 8509	2.652 3352	33
34	1.525 5663	1.658 9964	1.960 6760	2.315 3221	2.731 9053	34
35	1.544 6359	1.683 8813	1.999 8896	2.373 2052	2.813 8625	35
36	1.563 9438	1.709 1395	2.039 8873	2.432 5353	2.898 2783	36
37	1.583 4931	1.734 7766	2.080 6851	2.493 3487	2.985 2267	37
38	1.603 2868	1.760 7983	2.122 2988	2.555 6824	3.074 7835	38
39	1.623 3279	1.787 2102	2.164 7448	2.619 5745	3.167 0270	39
40	1.643 6195	1.814 0184	2.208 0397	2.685 0638	3.262 0378	40
41	1.664 1647	1.841 2287	2.252 2005	2.752 1904	3.359 8989	41
42	1.684 9668	1.868 8471	2.297 2445	2.820 9952	3.460 6959	42
43	1.706 0289	1.896 8798	2.343 1894	2.891 5201	3.564 5168	43
44	1.727 3542	1.925 3330	2.390 0531	2.963 8081	3.671 4523	44
45	1.748 9461	1.954 2130	2.437 8542	3.037 9033	3.781 5958	45
46	1.770 8080	1.983 5262	2.486 6113	3.113 8509	3.895 0437	46
47	1.792 9431	2.013 2791	2.536 3435	3.191 6971	4.011 8950	47
48	1.815 3549	2.043 4783	2.587 0704	3.271 4896	4.132 2519	48
49	1.838 0468	2.074 1305	2.638 8118	3.353 2768	4.256 2194	49
50	1.861 0224	2.105 2424	2.691 5880	3.437 1087	4.383 9060	50

COMPOUND AMOUNT (A) OF $1

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
1	1.035 0000	1.040 0000	1.045 0000	1.050 0000	1.060 0000	1
2	1.071 2250	1.081 6000	1.092 0250	1.102 5000	1.123 6000	2
3	1.108 7179	1.124 8640	1.141 1661	1.157 6250	1.191 0160	3
4	1.147 5230	1.169 8586	1.192 5186	1.215 5063	1.262 4770	4
5	1.187 6863	1.216 6529	1.246 1819	1.276 2816	1.338 2256	5
6	1.229 2553	1.265 3190	1.302 2601	1.340 0956	1.418 5191	6
7	1.272 2793	1.315 9318	1.360 8618	1.407 1004	1.503 6303	7
8	1.316 8090	1.368 5691	1.422 1006	1.477 4554	1.593 8481	8
9	1.362 8974	1.423 3118	1.486 0951	1.551 3282	1.689 4790	9
10	1.410 5988	1.480 2443	1.552 9694	1.628 8946	1.790 8477	10
11	1.459 9697	1.539 4541	1.622 8530	1.710 3394	1.898 2986	11
12	1.511 0687	1.601 0322	1.695 8814	1.795 8563	2.012 1965	12
13	1.563 9561	1.665 0735	1.772 1961	1.885 6491	2.132 9283	13
14	1.618 6945	1.731 6764	1.851 9449	1.979 9316	2.260 9040	14
15	1.675 3488	1.800 9435	1.935 2824	2.078 9282	2.396 5582	15
16	1.733 9860	1.872 9812	2.022 3702	2.182 8746	2.540 3517	16
17	1.794 6756	1.947 9005	2.113 3768	2.292 0183	2.692 7728	17
18	1.857 4892	2.025 8165	2.208 4788	2.406 6192	2.854 3392	18
19	1.922 5013	2.106 8492	2.307 8603	2.526 9502	3.025 5995	19
20	1.989 7889	2.191 1231	2.411 7140	2.653 2977	3.207 1355	20
21	2.059 4315	2.278 7681	2.520 2412	2.785 9626	3.399 5636	21
22	2.131 5116	2.369 9188	2.633 6520	2.925 2607	3.603 5374	22
23	2.206 1145	2.464 7155	2.752 1663	3.071 5238	3.819 7497	23
24	2.283 3285	2.563 3042	2.876 0138	3.225 0999	4.048 9346	24
25	2.363 2450	2.665 8363	3.005 4345	3.386 3549	4.291 8707	25
26	2.445 9586	2.772 4698	3.140 6790	3.555 6727	4.549 3830	26
27	2.531 5671	2.883 3686	3.282 0096	3.733 4563	4.822 3459	27
28	2.620 1720	2.998 7033	3.429 7000	3.920 1291	5.111 6867	28
29	2.711 8780	3.118 6515	3.584 0365	4.116 1356	5.418 3879	29
30	2.806 7937	3.243 3975	3.745 3181	4.321 9424	5.743 4912	30
31	2.905 0315	3.373 1334	3.913 8575	4.538 0395	6.088 1006	31
32	3.006 7076	3.508 0587	4.089 9810	4.764 9415	6.453 3867	32
33	3.111 9424	3.648 3811	4.274 0302	5.003 1885	6.840 5899	33
34	3.220 8603	3.794 3163	4.466 3615	5.253 3480	7.251 0253	34
35	3.333 5904	3.946 0890	4.667 3478	5.516 0154	7.686 0868	35
36	3.450 2661	4.103 9326	4.877 3785	5.791 8161	8.147 2520	36
37	3.571 0254	4.268 0899	5.096 8605	6.081 4069	8.636 0871	37
38	3.696 0113	4.438 8135	5.326 2192	6.385 4773	9.154 2523	38
39	3.825 3717	4.616 3660	5.565 8991	6.704 7512	9.703 5075	39
40	3.959 2597	4.801 0206	5.816 3645	7.039 9887	10.285 7179	40
41	4.097 8338	4.993 0615	6.078 1009	7.391 9881	10.902 8610	41
42	4.241 2580	5.192 7839	6.351 6155	7.761 5876	11.557 0327	42
43	4.389 7020	5.400 4953	6.637 4382	8.149 6669	12.250 4546	43
44	4.543 3416	5.616 5151	6.936 1229	8.557 1503	12.985 4819	44
45	4.702 3586	5.841 1757	7.248 2484	8.985 0078	13.764 6108	45
46	4.866 9411	6.074 8227	7.574 4196	9.434 2582	14.590 4875	46
47	5.037 2840	6.317 8156	7.915 2685	9.905 9711	15.465 9167	47
48	5.213 5890	6.570 5282	8.271 4556	10.401 2696	16.393 8717	48
49	5.396 0646	6.833 3494	8.643 6711	10.921 3331	17.377 5040	49
50	5.584 9269	7.106 6833	9.032 6363	11.467 3998	18.420 1543	50

PRESENT VALUE (P) OF $1

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	0.987 6543	0.985 2217	0.980 3922	0.975 6098	0.970 8738	1
2	0.975 4611	0.970 6617	0.961 1688	0.951 8144	0.942 5959	2
3	0.963 4183	0.956 3170	0.942 3223	0.928 5994	0.915 1417	3
4	0.951 5243	0.942 1842	0.923 8454	0.905 9506	0.888 4870	4
5	0.939 7771	0.928 2603	0.905 7308	0.883 8543	0.862 6088	5
6	0.928 1749	0.914 5422	0.887 9714	0.862 2969	0.837 4843	6
7	0.916 7159	0.901 0268	0.870 5602	0.841 2652	0.813 0915	7
8	0.905 3985	0.887 7111	0.853 4904	0.820 7466	0.789 4092	8
9	0.894 2207	0.874 5922	0.836 7553	0.800 7284	0.766 4167	9
10	0.883 1809	0.861 6672	0.820 3483	0.781 1984	0.744 0939	10
11	0.872 2775	0.848 9332	0.804 2630	0.762 1448	0.722 4213	11
12	0.861 5086	0.836 3874	0.788 4932	0.743 5559	0.701 3799	12
13	0.850 8727	0.824 0270	0.773 0325	0.725 4204	0.680 9513	13
14	0.840 3681	0.811 8493	0.757 8750	0.707 7272	0.661 1178	14
15	0.829 9932	0.799 8515	0.743 0147	0.690 4656	0.641 8619	15
16	0.819 7463	0.788 0310	0.728 4458	0.673 6249	0.623 1669	16
17	0.809 6260	0.776 3853	0.714 1626	0.657 1951	0.605 0164	17
18	0.799 6306	0.764 9116	0.700 1594	0.641 1659	0.587 3946	18
19	0.789 7587	0.753 6075	0.686 4308	0.625 5277	0.570 2860	19
20	0.780 0085	0.742 4704	0.672 9713	0.610 2709	0.553 6758	20
21	0.770 3788	0.731 4980	0.659 7758	0.595 3863	0.537 5493	21
22	0.760 8680	0.720 6876	0.646 8390	0.580 8647	0.521 8925	22
23	0.751 4745	0.710 0371	0.634 1559	0.566 6972	0.506 6917	23
24	0.742 1971	0.699 5439	0.621 7215	0.552 8754	0.491 9337	24
25	0.733 0341	0.689 2058	0.609 5309	0.539 3906	0.477 6056	25
26	0.723 9843	0.679 0205	0.597 5793	0.526 2347	0.463 6947	26
27	0.715 0463	0.668 9857	0.585 8620	0.513 3997	0.450 1891	27
28	0.706 2185	0.659 0993	0.574 3746	0.500 8778	0.437 0768	28
29	0.697 4998	0.649 3589	0.563 1123	0.488 6613	0.424 3464	29
30	0.688 8887	0.639 7624	0.552 0709	0.476 7427	0.411 9868	30
31	0.680 3839	0.630 3078	0.541 2460	0.465 1148	0.399 9871	31
32	0.671 9841	0.620 9929	0.530 6333	0.453 7705	0.388 3370	32
33	0.663 6880	0.611 8157	0.520 2287	0.442 7030	0.377 0262	33
34	0.655 4943	0.602 7741	0.510 0282	0.431 9053	0.366 0449	34
35	0.647 4018	0.593 8661	0.500 0276	0.421 3711	0.355 3834	35
36	0.639 4092	0.585 0897	0.490 2232	0.411 0937	0.345 0324	36
37	0.631 5152	0.576 4431	0.480 6109	0.401 0670	0.334 9829	37
38	0.623 7187	0.567 9242	0.471 1872	0.391 2849	0.325 2262	38
39	0.616 0185	0.559 5313	0.461 9482	0.381 7414	0.315 7535	39
40	0.608 4133	0.551 2623	0.452 8904	0.372 4306	0.306 5568	40
41	0.600 9021	0.543 1156	0.444 0102	0.363 3469	0.297 6280	41
42	0.593 4835	0.535 0893	0.435 3041	0.354 4848	0.288 9592	42
43	0.586 1566	0.527 1815	0.426 7688	0.345 8389	0.280 5429	43
44	0.578 9201	0.519 3907	0.418 4007	0.337 4038	0.272 3718	44
45	0.571 7729	0.511 7149	0.410 1968	0.329 1744	0.264 4386	45
46	0.564 7140	0.504 1527	0.402 1537	0.321 1458	0.256 7365	46
47	0.557 7422	0.496 7021	0.394 2684	0.313 3129	0.249 2588	47
48	0.550 8565	0.489 3617	0.386 5376	0.305 6712	0.241 9988	48
49	0.544 0558	0.482 1298	0.378 9584	0.298 2158	0.234 9503	49
50	0.537 3391	0.475 0047	0.371 5279	0.290 9422	0.228 0171	50

PRESENT VALUE (P) OF $1

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
1	0.966 1836	0.961 5385	0.956 9378	0.952 3810	0.943 3962	1
2	0.933 5107	0.924 5562	0.915 7300	0.907 0295	0.889 9964	2
3	0.901 9427	0.888 9964	0.876 2966	0.863 8376	0.839 6193	3
4	0.871 4422	0.854 8042	0.838 5613	0.822 7025	0.792 0937	4
5	0.841 9732	0.821 9271	0.802 4510	0.783 5262	0.747 2582	5
6	0.813 5006	0.790 3145	0.767 8957	0.746 2154	0.704 9605	6
7	0.785 9910	0.759 9178	0.734 8285	0.710 6813	0.665 0571	7
8	0.759 4116	0.730 6902	0.703 1851	0.676 8394	0.627 4124	8
9	0.733 7310	0.702 5867	0.672 9044	0.644 6089	0.591 8985	9
10	0.708 9188	0.675 5642	0.643 9277	0.613 9133	0.558 3948	10
11	0.684 9457	0.649 5809	0.616 1987	0.584 6793	0.526 7875	11
12	0.661 7833	0.624 5970	0.589 6639	0.556 8374	0.496 9694	12
13	0.639 4042	0.600 5741	0.564 2716	0.530 3214	0.468 8390	13
14	0.617 7818	0.577 4751	0.539 9729	0.505 0680	0.442 3010	14
15	0.596 8906	0.555 2645	0.516 7204	0.481 0171	0.417 2651	15
16	0.576 7059	0.533 9082	0.494 4693	0.458 1115	0.393 6463	16
17	0.557 2038	0.513 3732	0.473 1764	0.436 2967	0.371 3644	17
18	0.538 3611	0.493 6281	0.452 8004	0.415 5207	0.350 3438	18
19	0.520 1557	0.474 6424	0.433 3018	0.395 7340	0.330 5130	19
20	0.502 5659	0.456 3869	0.414 6429	0.376 8895	0.311 8047	20
21	0.485 5709	0.438 8336	0.396 7874	0.358 9424	0.294 1554	21
22	0.469 1506	0.421 9554	0.379 7009	0.341 8499	0.277 5051	22
23	0.453 2856	0.405 7263	0.363 3501	0.325 5713	0.261 7973	23
24	0.437 9571	0.390 1215	0.347 7035	0.310 0679	0.246 9785	24
25	0.423 1470	0.375 1168	0.332 7306	0.295 3028	0.232 9986	25
26	0.408 8377	0.360 6892	0.318 4025	0.281 2407	0.219 8100	26
27	0.395 0122	0.346 8166	0.304 6914	0.267 8483	0.207 3679	27
28	0.381 6543	0.333 4775	0.291 5707	0.255 0936	0.195 6301	28
29	0.368 7482	0.320 6514	0.279 0150	0.242 9463	0.184 5567	29
30	0.356 2784	0.308 3187	0.267 0000	0.231 3774	0.174 1101	30
31	0.344 2303	0.296 4603	0.255 5024	0.220 3595	0.164 2548	31
32	0.332 5897	0.285 0579	0.244 4999	0.209 8662	0.154 9574	32
33	0.321 3427	0.274 0942	0.233 9712	0.199 8725	0.146 1862	33
34	0.310 4761	0.263 5521	0.223 8959	0.190 3548	0.137 9115	34
35	0.299 9769	0.253 4155	0.214 2544	0.181 2903	0.130 1052	35
36	0.289 8327	0.243 6687	0.205 0282	0.172 6574	0.122 7408	36
37	0.280 0316	0.234 2968	0.196 1992	0.164 4356	0.115 7932	37
38	0.270 5619	0.225 2854	0.187 7504	0.156 6054	0.109 2389	38
39	0.261 4125	0.216 6206	0.179 6655	0.149 1480	0.103 0555	39
40	0.252 5725	0.208 2890	0.171 9287	0.142 0457	0.097 2222	40
41	0.244 0314	0.200 2779	0.164 5251	0.135 2816	0.091 7190	41
42	0.235 7791	0.192 5749	0.157 4403	0.128 8396	0.086 5274	42
43	0.227 8059	0.185 1682	0.150 6605	0.122 7044	0.081 6296	43
44	0.220 1023	0.178 0463	0.144 1728	0.116 8613	0.077 0091	44
45	0.212 6592	0.171 1984	0.137 9644	0.111 2965	0.072 6501	45
46	0.205 4679	0.164 6139	0.132 0233	0.105 9967	0.068 5378	46
47	0.198 5197	0.158 2826	0.126 3381	0.100 9492	0.064 6583	47
48	0.191 8065	0.152 1948	0.120 8977	0.096 1421	0.060 9984	48
49	0.185 3202	0.146 3411	0.115 6916	0.091 5639	0.057 5457	49
50	0.179 0534	0.140 7126	0.110 7096	0.087 2037	0.054 2884	50

AMOUNT ($A_{\overline{n}|}$) OF $1 PER ANNUM

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1
2	2.012 5000	2.015 0000	2.020 0000	2.025 0000	2.030 0000	2
3	3.037 6562	3.045 2250	3.060 4000	3.075 6250	3.090 9000	3
4	4.075 6269	4.090 9034	4.121 6080	4.152 5156	4.183 6270	4
5	5.126 5723	5.152 2669	5.204 0402	5.256 3285	5.309 1358	5
6	6.190 6544	6.229 5509	6.308 1210	6.387 7367	6.468 4099	6
7	7.268 0376	7.322 9942	7.434 2834	7.547 4301	7.662 4622	7
8	8.358 8881	8.432 8391	8.582 9691	8.736 1159	8.892 3360	8
9	9.463 3742	9.559 3317	9.754 6284	9.954 5188	10.159 1061	9
10	10.581 6664	10.702 7217	10.949 7210	11.203 3818	11.463 8793	10
11	11.713 9372	11.863 2625	12.168 7154	12.483 4663	12.807 7957	11
12	12.860 3614	13.041 2114	13.412 0897	13.795 5530	14.192 0296	12
13	14.021 1159	14.236 8296	14.680 3315	15.140 4418	15.617 7904	13
14	15.196 3799	15.450 3820	15.973 9382	16.518 9528	17.086 3242	14
15	16.386 3346	16.682 1378	17.293 4169	17.931 9267	18.598 9139	15
16	17.591 1638	17.932 3698	18.639 2853	19.380 2248	20.156 8813	16
17	18.811 0534	19.201 3554	20.012 0710	20.864 7304	21.761 5877	17
18	20.046 1915	20.489 3757	21.412 3124	22.386 3487	23.414 4354	18
19	21.296 7689	21.796 7164	22.840 5586	23.946 0074	25.116 8684	19
20	22.562 9785	23.123 6671	24.297 3698	25.544 6576	26.870 3745	20
21	23.845 0158	24.470 5221	25.783 3172	27.183 2741	28.676 4857	21
22	25.143 0785	25.837 5799	27.298 9835	28.862 8559	30.536 7803	22
23	26.457 3669	27.225 1436	28.844 9632	30.584 4273	32.452 8837	23
24	27.788 0840	28.633 5208	30.421 8625	32.349 0380	34.426 4702	24
25	29.135 4351	30.063 0236	32.030 2997	34.157 7639	36.459 2643	25
26	30.499 6280	31.513 9690	33.670 9057	36.011 7080	38.553 0423	26
27	31.880 8734	32.986 6785	35.344 3238	37.912 0007	40.709 6335	27
28	33.279 3843	34.481 4787	37.051 2103	39.859 8007	42.930 9225	28
29	34.695 3766	35.998 7009	38.792 2345	41.856 2958	45.218 8502	29
30	36.129 0688	37.538 6814	40.568 0792	43.902 7032	47.575 4157	30
31	37.580 6822	39.101 7616	42.379 4408	46.000 2707	50.002 6782	31
32	39.050 4407	40.688 2880	44.227 0296	48.150 2775	52.502 7585	32
33	40.538 5712	42.298 6123	46.111 5702	50.354 0344	55.077 8413	33
34	42.045 3033	43.933 0915	48.033 8016	52.612 8853	57.730 1765	34
35	43.570 8696	45.592 0879	49.994 4776	54.928 2074	60.462 0818	35
36	45.115 5055	47.275 9692	51.994 3672	57.301 4126	63.275 9443	36
37	46.679 4493	48.985 1087	54.034 2545	59.733 9479	66.174 2226	37
38	48.262 9424	50.719 8854	56.114 9396	62.227 2966	69.159 4493	38
39	49.866 2292	52.480 6837	58.237 2384	64.782 9791	72.234 2328	39
40	51.489 5571	54.267 8939	60.401 9832	67.402 5535	75.401 2597	40
41	53.133 1765	56.081 9123	62.610 0228	70.087 6174	78.663 2975	41
42	54.797 3412	57.923 1410	64.862 2233	72.839 8078	82.023 1965	42
43	56.482 3080	59.791 9881	67.159 4678	75.660 8030	85.483 8923	43
44	58.188 3369	61.688 8679	69.502 6571	78.552 3231	89.048 4091	44
45	59.915 6911	63.614 2010	71.892 7103	81.516 1312	92.719 8614	45
46	61.664 6372	65.568 4140	74.330 5645	84.554 0344	96.501 4572	46
47	63.435 4452	67.551 9402	76.817 1758	87.667 8853	100.396 5009	47
48	65.228 3884	69.565 2193	79.353 5193	90.859 5824	104.408 3960	48
49	67.043 7431	71.608 6976	81.940 5897	94.131 0720	108.540 6479	49
50	68.881 7899	73.682 8280	84.579 4015	97.484 3288	112.796 8673	50

AMOUNT ($A_{\overline{n}|}$) OF $1 PER ANNUM

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1
2	2.035 0000	2.040 0000	2.045 0000	2.050 0000	2.060 0000	2
3	3.106 2250	3.121 6000	3.137 0250	3.152 5000	3.183 6000	3
4	4.214 9429	4.246 4640	4.278 1911	4.310 1250	4.374 6160	4
5	5.362 4659	5.416 3226	5.470 7097	5.525 6312	5.637 0930	5
6	6.550 1522	6.632 9755	6.716 8917	6.801 9128	6.975 3185	6
7	7.779 4075	7.898 2945	8.019 1518	8.142 0085	8.393 8376	7
8	9.051 6868	9.214 2263	9.380 0136	9.549 1089	9.897 4679	8
9	10.368 4958	10.582 7953	10.802 1142	11.026 5643	11.491 3160	9
10	11.731 3932	12.006 1071	12.288 2094	12.577 8925	13.180 7949	10
11	13.141 9919	13.486 3514	13.841 1788	14.206 7872	14.971 6426	11
12	14.601 9616	15.025 8055	15.464 0318	15.917 1265	16.869 9412	12
13	16.113 0303	16.626 8377	17.159 9132	17.712 9828	18.882 1377	13
14	17.676 9864	18.291 9112	18.932 0194	19.598 6320	21.015 0659	14
15	19.295 6809	20.023 5876	20.784 0543	21.578 5636	23.275 9699	15
16	20.971 0297	21.824 5311	22.719 3367	23.657 4918	25.672 5281	16
17	22.705 0157	23.697 5124	24.741 7069	25.840 3664	28.212 8798	17
18	24.499 6913	25.645 4129	26.855 0837	28.132 3847	30.905 6525	18
19	26.357 1805	27.671 2294	29.063 5625	30.539 0039	33.759 9917	19
20	28.279 6818	29.778 0786	31.371 4228	33.065 9541	36.785 5912	20
21	30.269 4707	31.969 2017	33.783 1368	35.719 2518	39.992 7267	21
22	32.328 9021	34.247 9698	36.303 3780	38.505 2144	43.392 2903	22
23	34.460 4137	36.617 8886	38.937 0300	41.430 4751	46.995 8277	23
24	36.666 5282	39.082 6041	41.689 1963	44.501 9989	50.815 5774	24
25	38.949 8567	41.645 9083	44.565 2101	47.727 0988	54.864 5120	25
26	41.313 1017	44.311 7446	47.570 6446	51.113 4538	59.156 3827	26
27	43.759 0602	47.084 2144	50.711 3236	54.669 1264	63.705 7657	27
28	46.290 6273	49.967 5830	53.993 3332	58.402 5828	68.528 1116	28
29	48.910 7993	52.966 2863	57.423 0332	62.322 7119	73.639 7983	29
30	51.622 6773	56.084 9377	61.007 0697	66.438 8475	79.058 1862	30
31	54.429 4710	59.328 3353	64.752 3878	70.760 7899	84.801 6774	31
32	57.334 5025	62.701 4687	68.666 2452	75.298 8294	90.889 7780	32
33	60.341 2101	66.209 5274	72.756 2263	80.063 7708	97.343 1647	33
34	63.453 1524	69.857 9085	77.030 2565	85.066 9594	104.183 7546	34
35	66.674 0127	73.652 2249	81.496 6180	90.320 3074	111.434 7799	35
36	70.007 6032	77.598 3138	86.163 9658	95.836 3227	119.120 8667	36
37	73.457 8693	81.702 2464	91.041 3443	101.628 1389	127.268 1187	37
38	77.028 8947	85.970 3363	96.138 2048	107.709 5458	135.904 2058	38
39	80.724 9060	90.409 1497	101.464 4240	114.095 0231	145.058 4581	39
40	84.550 2777	95.025 5157	107.030 3231	120.799 7742	154.761 9656	40
41	88.509 5375	99.826 5363	112.846 6876	127.839 7630	165.047 6836	41
42	92.607 3713	104.819 5978	118.924 7885	135.231 7511	175.950 5446	42
43	96.848 6293	110.012 3817	125.276 4040	142.993 3387	187.507 5772	43
44	101.238 3313	115.412 8770	131.913 8422	151.143 0056	199.758 0319	44
45	105.781 6729	121.029 3920	138.849 9651	159.700 1559	212.743 5138	45
46	110.484 0314	126.870 5677	146.098 2135	168.685 1637	226.508 1246	46
47	115.350 9725	132.945 3904	153.672 6331	178.119 4218	241.098 6121	47
48	120.388 2566	139.263 2060	161.587 9016	188.025 3929	256.564 5288	48
49	125.601 8456	145.833 7343	169.859 3572	198.426 6626	272.958 4005	49
50	130.997 9102	152.667 0837	178.503 0283	209.347 9957	290.335 9046	50

PRESENT VALUE ($P_{\overline{n}|}$) OF $1 PER ANNUM

n	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	0.987 6543	0.985 2217	0.980 3922	0.975 6098	0.970 8738	1
2	1.963 1154	1.955 8834	1.941 5609	1.927 4242	1.913 4697	2
3	2.926 5307	2.912 2004	2.883 8833	2.856 0236	2.828 6114	3
4	3.878 0580	3.854 3846	3.807 7287	3.761 9742	3.717 0984	4
5	4.817 8350	4.782 6450	4.713 4595	4.645 8285	4.579 7072	5
6	5.746 0099	5.697 1872	5.601 4309	5.508 1254	5.417 1914	6
7	6.662 7258	6.598 2140	6.471 9911	6.349 3906	6.230 2830	7
8	7.568 1243	7.485 9251	7.325 4814	7.170 1372	7.019 6922	8
9	8.462 3450	8.360 5173	8.162 2367	7.970 8655	7.786 1089	9
10	9.345 5259	9.222 1846	8.982 5850	8.752 0639	8.530 2028	10
11	10.217 8034	10.071 1178	9.786 8480	9.514 2087	9.252 6241	11
12	11.079 3120	10.907 5052	10.575 3412	10.257 7646	9.954 0040	12
13	11.930 1847	11.731 5322	11.348 3737	10.983 1850	10.634 9553	13
14	12.770 5527	12.543 3815	12.106 2488	11.690 9122	11.296 0731	14
15	13.600 5459	13.343 2330	12.849 2635	12.381 3777	11.937 9351	15
16	14.420 2923	14.131 2641	13.577 7093	13.055 0027	12.561 1020	16
17	15.229 9183	14.907 6493	14.291 8719	13.712 1977	13.166 1185	17
18	16.029 5489	15.672 5609	14.992 0312	14.353 3636	13.753 5131	18
19	16.819 3076	16.426 1684	15.678 4620	14.978 8913	14.323 7991	19
20	17.599 3161	17.168 6388	16.351 4333	15.589 1623	14.877 4749	20
21	18.369 6949	17.900 1367	17.011 2092	16.184 5486	15.415 0241	21
22	19.130 5629	18.620 8244	17.658 0482	16.765 4132	15.936 9166	22
23	19.882 0374	19.330 8614	18.292 2041	17.332 1105	16.443 6084	23
24	20.624 2345	20.030 4054	18.913 9256	17.884 9858	16.935 5421	24
25	21.357 2686	20.719 6112	19.523 4565	18.424 3764	17.413 1477	25
26	22.081 2530	21.398 6317	20.121 0358	18.950 6111	17.876 8424	26
27	22.796 2993	22.067 6175	20.706 8978	19.464 0109	18.327 0315	27
28	23.502 5178	22.726 7167	21.281 2724	19.964 8887	18.764 1082	28
29	24.200 0176	23.376 0756	21.844 3847	20.453 5499	19.188 4546	29
30	24.888 9062	24.015 8380	22.396 4556	20.930 2926	19.600 4413	30
31	25.569 2901	24.646 1458	22.937 7015	21.395 4074	20.000 4285	31
32	26.241 2742	25.267 1387	23.468 3348	21.849 1780	20.388 7655	32
33	26.904 9621	25.878 9544	23.988 5636	22.291 8809	20.765 7918	33
34	27.560 4564	26.481 7285	24.498 5917	22.723 7863	21.131 8367	34
35	28.207 8582	27.075 5946	24.998 6193	23.145 1573	21.487 2201	35
36	28.847 2674	27.660 6843	25.488 8425	23.556 2511	21.832 2525	36
37	29.478 7826	28.237 1274	25.969 4534	23.957 3181	22.167 2354	37
38	30.102 5013	28.805 0516	26.440 6406	24.348 6030	22.492 4616	38
39	30.718 5198	29.364 5829	26.902 5888	24.730 3444	22.808 2151	39
40	31.326 9332	29.915 8452	27.355 4792	25.102 7750	23.114 7720	40
41	31.927 8352	30.458 9608	27.799 4895	25.466 1220	23.412 4000	41
42	32.521 3187	30.994 0500	28.234 7936	25.820 6068	23.701 3592	42
43	33.107 4753	31.521 2316	28.661 5623	26.166 4457	23.981 9021	43
44	33.686 3954	32.040 6222	29.079 9631	26.503 8495	24.254 2739	44
45	34.258 1682	32.552 3372	29.490 1599	26.833 0239	24.518 7125	45
46	34.822 8822	33.056 4898	29.892 3136	27.154 1696	24.775 4491	46
47	35.380 6244	33.553 1919	30.286 5820	27.467 4826	25.024 7078	47
48	35.931 4809	34.042 5536	30.673 1196	27.773 1537	25.266 7066	48
49	36.475 5367	34.524 6834	31.052 0780	28.071 3695	25.501 6569	49
50	37.012 8757	34.999 6881	31.423 6059	28.362 3117	25.729 7640	50

PRESENT VALUE ($P_{\overline{n}|}$) OF $1 PER ANNUM

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	n
1	0.966 1836	0.961 5385	0.956 9378	0.952 3810	0.943 3962	1
2	1.899 6943	1.886 0947	1.872 6678	1.859 4104	1.833 3927	2
3	2.801 6370	2.775 0910	2.748 9644	2.723 2480	2.673 0119	3
4	3.673 0792	3.629 8952	3.587 5257	3.545 9505	3.465 1056	4
5	4.515 0524	4.451 8223	4.389 9767	4.329 4767	4.212 3638	5
6	5.328 5530	5.242 1369	5.157 8725	5.075 6921	4.917 3243	6
7	6.114 5440	6.002 0547	5.892 7009	5.786 3734	5.582 3814	7
8	6.873 9555	6.732 7449	6.595 8861	6.463 2128	6.209 7938	8
9	7.607 6865	7.435 3316	7.268 7905	7.107 8217	6.801 6923	9
10	8.316 6053	8.110 8958	7.912 7182	7.721 7349	7.360 0871	10
11	9.001 5510	8.760 4767	8.528 9169	8.306 4142	7.886 8746	11
12	9.663 3343	9.385 0738	9.118 5808	8.863 2516	8.383 8439	12
13	10.302 7385	9.985 6478	9.682 8524	9.393 5730	8.852 6830	13
14	10.920 5203	10.563 1229	10.222 8253	9.898 6409	9.294 9839	14
15	11.517 4109	11.118 3874	10.739 5457	10.379 6580	9.712 2490	15
16	12.094 1168	11.652 2956	11.234 0150	10.837 7696	10.105 8953	16
17	12.651 3206	12.165 6689	11.707 1914	11.274 0662	10.477 2597	17
18	13.189 6817	12.659 2970	12.159 9918	11.689 5869	10.827 6035	18
19	13.709 8374	13.133 9394	12.593 2936	12.085 3209	11.158 1165	19
20	14.212 4033	13.590 3263	13.007 9365	12.462 2103	11.469 9212	20
21	14.697 9742	14.029 1599	13.404 7239	12.821 1527	11.764 0766	21
22	15.167 1248	14.451 1153	13.784 4248	13.163 0026	12.041 5817	22
23	15.620 4105	14.856 8417	14.147 7749	13.488 5739	12.303 3790	23
24	16.058 3676	15.246 9631	14.495 4784	13.798 6418	12.550 3575	24
25	16.481 5146	15.622 0799	14.828 2090	14.093 9446	12.783 3562	25
26	16.890 3523	15.982 7692	15.146 6114	14.375 1853	13.003 1662	26
27	17.285 3645	16.329 5857	15.451 3028	14.643 0336	13.210 5341	27
28	17.667 0188	16.663 0632	15.742 8735	14.898 1273	13.406 1643	28
29	18.035 7670	16.983 7146	16.021 8885	15.141 0736	13.590 7210	29
30	18.392 0454	17.292 0333	16.288 8885	15.372 4510	13.764 8312	30
31	18.736 2758	17.588 4936	16.544 3910	15.592 8105	13.929 0860	31
32	19.068 8655	17.873 5515	16.788 8909	15.802 6767	14.084 0434	32
33	19.390 2082	18.147 6457	17.022 8621	16.002 5492	14.230 2296	33
34	19.700 6842	18.411 1978	17.246 7580	16.192 9040	14.368 1411	34
35	20.000 6611	18.664 6132	17.461 0124	16.374 1943	14.498 2464	35
36	20.290 4938	18.908 2820	17.666 0406	16.546 8517	14.620 9871	36
37	20.570 5254	19.142 5788	17.862 2398	16.711 2873	14.736 7803	37
38	20.841 0874	19.367 8642	18.049 9902	16.867 8927	14.846 0192	38
39	21.102 4999	19.584 4848	18.229 6557	17.017 0407	14.949 0747	39
40	21.355 0723	19.792 7739	18.401 5844	17.159 0864	15.046 2969	40
41	21.599 1037	19.993 0518	18.566 1095	17.294 3680	15.138 0159	41
42	21.834 8828	20.185 6267	18.723 5498	17.423 2076	15.224 5433	42
43	22.062 6887	20.370 7949	18.874 2103	17.545 9120	15.306 1729	43
44	22.282 7910	20.548 8413	19.018 3831	17.662 7733	15.383 1820	44
45	22.495 4503	20.720 0397	19.156 3474	17.774 0698	15.455 8321	45
46	22.700 9181	20.884 6536	19.288 3707	17.880 0665	15.524 3699	46
47	22.899 4378	21.042 9361	19.414 7088	17.981 0157	15.589 0282	47
48	23.091 2443	21.195 1309	19.535 6065	18.077 1578	15.650 0266	48
49	23.276 5645	21.341 4720	19.651 2981	18.168 7217	15.707 5723	49
50	23.455 6179	21.482 1846	19.762 0078	18.255 9255	15.761 8606	50

3. Logarithms of Numbers

Four-Place Logarithms, 1.00–9.99. The table on pages 314 and 315 gives to four places the common logarithms of the numbers 1.00–9.99.

In the table the characteristic of the logarithm, which is always zero, is omitted in each case.

The student is already familiar with the use of this table, both in finding the logarithm of a given number, as described on page 72, and in finding the antilogarithm of a given logarithm, as described on page 73.

The first two figures of the number are given in the column at the left, headed **N**, and the third figure is given in the columns headed **0** to **9** at the right of column **N**. The difference columns at the right, headed **1** to **9** in smaller type, indicate the number to be added in each case when the number whose logarithm is to be found contains a fourth figure. That is, this table can be used in finding directly the four-place logarithm of any number of four figures.

Instead of using the difference columns, the differences may be found by ordinary interpolation, as described on page 72. Since many tables of four-place logarithms are not provided with difference columns, as is also the case with the six-place table described on the following page, the student should be able to apply this method readily.

The characteristic of the logarithm of a number not given in the table is easily found by applying the rules on page 70.

Thus, since $26.5 = 2.65 \times 10^1$, the characteristic of log 26.5 is 1 greater than that of log 2.65; that is, $\log 26.5 = 1.4232$. In the same way, since $0.00265 = 2.65 \times 10^{-3}$, we have $\log 0.00265 = \bar{3}.4232$.

It should be remembered that no result obtained from a four-place table can be accurate to more than four figures, and frequently the fourth figure is only approximate.

For example, with logarithms we find that $0.0256^3 = 0.00001677$, while the result by actual multiplication is 0.000016777216. In practical work such differences are ordinarily negligible.

Six-Place Logarithms, 100.0–999.9. The table on pages 316–333 gives to six places the common logarithms of the numbers 100.0–999.9. This table should be used whenever the four-place table fails to give results that are sufficiently accurate.

In this table both the characteristic, which is always 2, and the decimal point are omitted in each case.

The first three figures of the number are given in the column at the left, headed **N**, and the fourth figure is given in the columns headed **0** to **9** at the right of column **N**. Unlike the four-place table, there are no difference columns so that in finding the logarithm of a number of more than four figures it is necessary to use ordinary interpolation, as described for the four-place table on page 72.

In the columns headed **1** to **9** only the last four figures of the mantissas are given, and to them should be prefixed (except as noted below) the first two of the six figures last given in the column headed **0**.

For example, on page 316 we find that $\log 103.0 = 2.012837$, and hence for $\log 103.1$, for which the table gives 3259, we have $\log 103.1 = 2.013259$.

In the cases where asterisks (*) are used, it means that the first two figures of such a mantissa are 1 greater than the two last given, the asterisk itself indicating a zero.

For example, from page 316, $\log 102.3 = 2.009876$ and $\log 102.4 = 2.010300$. Similarly, from the same page, $\log 117.4 = 2.069668$ and $\log 117.5 = 2.070038$.

The characteristic of the logarithm of a number not given in the table is easily found by applying the same rules as are used for four-place logarithms (page 70).

The student should study the following examples carefully in order to make sure that the method of using the six-place table is entirely clear:

(1) From page 328, $\log 0.0007 = \bar{4}.845098$.
(2) From page 320, $\log 3.091 = 0.490099$.
(3) From page 327, $\log 6705.5 = 3.826399 + .000033 = 3.826432$.
(4) From page 333, $\log 977{,}462 = 5.990072 + .000028 = 5.990100$.
(5) From page 327, $\text{antilog } 2.842141 = 695.2 + 0.050 = 695.250$.
(6) From page 325, $\text{antilog } \bar{1}.750244 = 0.5626 + 0.000057 = 0.562657$.

FOUR-PLACE LOGARITHMS. 1.00–5.49

N	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
1.0	.0000	.0043	.0086	.0128	.0170	.0212	.0253	.0294	.0334	.0374	4 8 12	17 21 25	29 33 37
1.1	.0414	.0453	.0492	.0531	.0569	.0607	.0645	.0682	.0719	.0755	4 8 11	15 19 23	26 30 34
1.2	.0792	.0828	.0864	.0899	.0934	.0969	.1004	.1038	.1072	.1106	3 7 10	14 17 21	24 28 31
1.3	.1139	.1173	.1206	.1239	.1271	.1303	.1335	.1367	.1399	.1430	3 6 10	13 16 19	23 26 29
1.4	.1461	.1492	.1523	.1553	.1584	.1614	.1644	.1673	.1703	.1732	3 6 9	12 15 18	21 24 27
1.5	.1761	.1790	.1818	.1847	.1875	.1903	.1931	.1959	.1987	.2014	3 6 8	11 14 17	20 22 25
1.6	.2041	.2068	.2095	.2122	.2148	.2175	.2201	.2227	.2253	.2279	3 5 8	11 13 16	18 21 24
1.7	.2304	.2330	.2355	.2380	.2405	.2430	.2455	.2480	.2504	.2529	2 5 7	10 12 15	17 20 22
1.8	.2553	.2577	.2601	.2625	.2648	.2672	.2695	.2718	.2742	.2765	2 5 7	9 12 14	16 19 21
1.9	.2788	.2810	.2833	.2856	.2878	.2900	.2923	.2945	.2967	.2989	2 4 7	9 11 13	16 18 20
2.0	.3010	.3032	.3054	.3075	.3096	.3118	.3139	.3160	.3181	.3201	2 4 6	8 11 13	15 17 19
2.1	.3222	.3243	.3263	.3284	.3304	.3324	.3345	.3365	.3385	.3404	2 4 6	8 10 12	14 16 18
2.2	.3424	.3444	.3464	.3483	.3502	.3522	.3541	.3560	.3579	.3598	2 4 6	8 10 12	14 15 17
2.3	.3617	.3636	.3655	.3674	.3692	.3711	.3729	.3747	.3766	.3784	2 4 6	7 9 11	13 15 17
2.4	.3802	.3820	.3838	.3856	.3874	.3892	.3909	.3927	.3945	.3962	2 4 5	7 9 11	12 14 16
2.5	.3979	.3997	.4014	.4031	.4048	.4065	.4082	.4099	.4116	.4133	2 3 5	7 9 10	12 14 15
2.6	.4150	.4166	.4183	.4200	.4216	.4232	.4249	.4265	.4281	.4298	2 3 5	7 8 10	11 13 15
2.7	.4314	.4330	.4346	.4362	.4378	.4393	.4409	.4425	.4440	.4456	2 3 5	6 8 9	11 13 14
2.8	.4472	.4487	.4502	.4518	.4533	.4548	.4564	.4579	.4594	.4609	2 3 5	6 8 9	11 12 14
2.9	.4624	.4639	.4654	.4669	.4683	.4698	.4713	.4728	.4742	.4757	1 3 4	6 7 9	10 12 13
3.0	.4771	.4786	.4800	.4814	.4829	.4843	.4857	.4871	.4886	.4900	1 3 4	6 7 9	10 11 13
3.1	.4914	.4928	.4942	.4955	.4969	.4983	.4997	.5011	.5024	.5038	1 3 4	6 7 8	10 11 12
3.2	.5051	.5065	.5079	.5092	.5105	.5119	.5132	.5145	.5159	.5172	1 3 4	5 7 8	9 11 12
3.3	.5185	.5198	.5211	.5224	.5237	.5250	.5263	.5276	.5289	.5302	1 3 4	5 6 8	9 10 12
3.4	.5315	.5328	.5340	.5353	.5366	.5378	.5391	.5403	.5416	.5428	1 3 4	5 6 8	9 10 11
3.5	.5441	.5453	.5465	.5478	.5490	.5502	.5514	.5527	.5539	.5551	1 2 4	5 6 7	9 10 11
3.6	.5563	.5575	.5587	.5599	.5611	.5623	.5635	.5647	.5658	.5670	1 2 4	5 6 7	8 10 11
3.7	.5682	.5694	.5705	.5717	.5729	.5740	.5752	.5763	.5775	.5786	1 2 3	5 6 7	8 9 10
3.8	.5798	.5809	.5821	.5832	.5843	.5855	.5866	.5877	.5888	.5899	1 2 3	5 6 7	8 9 10
3.9	.5911	.5922	.5933	.5944	.5955	.5966	.5977	.5988	.5999	.6010	1 2 3	4 5 7	8 9 10
4.0	.6021	.6031	.6042	.6053	.6064	.6075	.6085	.6096	.6107	.6117	1 2 3	4 5 6	8 9 10
4.1	.6128	.6138	.6149	.6160	.6170	.6180	.6191	.6201	.6212	.6222	1 2 3	4 5 6	7 8 9
4.2	.6232	.6243	.6253	.6263	.6274	.6284	.6294	.6304	.6314	.6325	1 2 3	4 5 6	7 8 9
4.3	.6335	.6345	.6355	.6365	.6375	.6385	.6395	.6405	.6415	.6425	1 2 3	4 5 6	7 8 9
4.4	.6435	.6444	.6454	.6464	.6474	.6484	.6493	.6503	.6513	.6522	1 2 3	4 5 6	7 8 9
4.5	.6532	.6542	.6551	.6561	.6571	.6580	.6590	.6599	.6609	.6618	1 2 3	4 5 6	7 8 9
4.6	.6628	.6637	.6646	.6656	.6665	.6675	.6684	.6693	.6702	.6712	1 2 3	4 5 6	7 7 8
4.7	.6721	.6730	.6739	.6749	.6758	.6767	.6776	.6785	.6794	.6803	1 2 3	4 5 5	6 7 8
4.8	.6812	.6821	.6830	.6839	.6848	.6857	.6866	.6875	.6884	.6893	1 2 3	4 4 5	6 7 8
4.9	.6902	.6911	.6920	.6928	.6937	.6946	.6955	.6964	.6972	.6981	1 2 3	4 4 5	6 7 8
5.0	.6990	.6998	.7007	.7016	.7024	.7033	.7042	.7050	.7059	.7067	1 2 3	3 4 5	6 7 8
5.1	.7076	.7084	.7093	.7101	.7110	.7118	.7126	.7135	.7143	.7152	1 2 3	3 4 5	6 7 8
5.2	.7160	.7168	.7177	.7185	.7193	.7202	.7210	.7218	.7226	.7235	1 2 2	3 4 5	6 7 7
5.3	.7243	.7251	.7259	.7267	.7275	.7284	.7292	.7300	.7308	.7316	1 2 2	3 4 5	6 6 7
5.4	.7324	.7332	.7340	.7348	.7356	.7364	.7372	.7380	.7388	.7396	1 2 2	3 4 5	6 6 7

In each of the above logarithms the characteristic is 0.

FOUR-PLACE LOGARITHMS. 5.50–9.99

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5.5	.7404	.7412	.7419	.7427	.7435	.7443	.7451	.7459	.7466	.7474	1	2	2	3	4	5	5	6	7
5.6	.7482	.7490	.7497	.7505	.7513	.7520	.7528	.7536	.7543	.7551	1	2	2	3	4	5	5	6	7
5.7	.7559	.7566	.7574	.7582	.7589	.7597	.7604	.7612	.7619	.7627	1	2	2	3	4	5	5	6	7
5.8	.7634	.7642	.7649	.7657	.7664	.7672	.7679	.7686	.7694	.7701	1	1	2	3	4	4	5	6	7
5.9	.7709	.7716	.7723	.7731	.7738	.7745	.7752	.7760	.7767	.7774	1	1	2	3	4	4	5	6	7
6.0	.7782	.7789	.7796	.7803	.7810	.7818	.7825	.7832	.7839	.7846	1	1	2	3	4	4	5	6	6
6.1	.7853	.7860	.7868	.7875	.7882	.7889	.7896	.7903	.7910	.7917	1	1	2	3	4	4	5	6	6
6.2	.7924	.7931	.7938	.7945	.7952	.7959	.7966	.7973	.7980	.7987	1	1	2	3	3	4	5	6	6
6.3	.7993	.8000	.8007	.8014	.8021	.8028	.8035	.8041	.8048	.8055	1	1	2	3	3	4	5	5	6
6.4	.8062	.8069	.8075	.8082	.8089	.8096	.8102	.8109	.8116	.8122	1	1	2	3	3	4	5	5	6
6.5	.8129	.8136	.8142	.8149	.8156	.8162	.8169	.8176	.8182	.8189	1	1	2	3	3	4	5	5	6
6.6	.8195	.8202	.8209	.8215	.8222	.8228	.8235	.8241	.8248	.8254	1	1	2	3	3	4	5	5	6
6.7	.8261	.8267	.8274	.8280	.8287	.8293	.8299	.8306	.8312	.8319	1	1	2	3	3	4	5	5	6
6.8	.8325	.8331	.8338	.8344	.8351	.8357	.8363	.8370	.8376	.8382	1	1	2	3	3	4	4	5	6
6.9	.8388	.8395	.8401	.8407	.8414	.8420	.8426	.8432	.8439	.8445	1	1	2	2	3	4	4	5	6
7.0	.8451	.8457	.8463	.8470	.8476	.8482	.8488	.8494	.8500	.8506	1	1	2	2	3	4	4	5	6
7.1	.8513	.8519	.8525	.8531	.8537	.8543	.8549	.8555	.8561	.8567	1	1	2	2	3	4	4	5	5
7.2	.8573	.8579	.8585	.8591	.8597	.8603	.8609	.8615	.8621	.8627	1	1	2	2	3	4	4	5	5
7.3	.8633	.8639	.8645	.8651	.8657	.8663	.8669	.8675	.8681	.8686	1	1	2	2	3	4	4	5	5
7.4	.8692	.8698	.8704	.8710	.8716	.8722	.8727	.8733	.8739	.8745	1	1	2	2	3	4	4	5	5
7.5	.8751	.8756	.8762	.8768	.8774	.8779	.8785	.8791	.8797	.8802	1	1	2	2	3	3	4	5	5
7.6	.8808	.8814	.8820	.8825	.8831	.8837	.8842	.8848	.8854	.8859	1	1	2	2	3	3	4	5	5
7.7	.8865	.8871	.8876	.8882	.8887	.8893	.8899	.8904	.8910	.8915	1	1	2	2	3	3	4	4	5
7.8	.8921	.8927	.8932	.8938	.8943	.8949	.8954	.8960	.8965	.8971	1	1	2	2	3	3	4	4	5
7.9	.8976	.8982	.8987	.8993	.8998	.9004	.9009	.9015	.9020	.9025	1	1	2	2	3	3	4	4	5
8.0	.9031	.9036	.9042	.9047	.9053	.9058	.9063	.9069	.9074	.9079	1	1	2	2	3	3	4	4	5
8.1	.9085	.9090	.9096	.9101	.9106	.9112	.9117	.9122	.9128	.9133	1	1	2	2	3	3	4	4	5
8.2	.9138	.9143	.9149	.9154	.9159	.9165	.9170	.9175	.9180	.9186	1	1	2	2	3	3	4	4	5
8.3	.9191	.9196	.9201	.9206	.9212	.9217	.9222	.9227	.9232	.9238	1	1	2	2	3	3	4	4	5
8.4	.9243	.9248	.9253	.9258	.9263	.9269	.9274	.9279	.9284	.9289	1	1	2	2	3	3	4	4	5
8.5	.9294	.9299	.9304	.9309	.9315	.9320	.9325	.9330	.9335	.9340	1	1	2	2	3	3	4	4	5
8.6	.9345	.9350	.9355	.9360	.9365	.9370	.9375	.9380	.9385	.9390	1	1	2	2	3	3	4	4	5
8.7	.9395	.9400	.9405	.9410	.9415	.9420	.9425	.9430	.9435	.9440	0	1	1	2	2	3	3	4	4
8.8	.9445	.9450	.9455	.9460	.9465	.9469	.9474	.9479	.9484	.9489	0	1	1	2	2	3	3	4	4
8.9	.9494	.9499	.9504	.9509	.9513	.9518	.9523	.9528	.9533	.9538	0	1	1	2	2	3	3	4	4
9.0	.9542	.9547	.9552	.9557	.9562	.9566	.9571	.9576	.9581	.9586	0	1	1	2	2	3	3	4	4
9.1	.9590	.9595	.9600	.9605	.9609	.9614	.9619	.9624	.9628	.9633	0	1	1	2	2	3	3	4	4
9.2	.9638	.9643	.9647	.9652	.9657	.9661	.9666	.9671	.9675	.9680	0	1	1	2	2	3	3	4	4
9.3	.9685	.9689	.9694	.9699	.9703	.9708	.9713	.9717	.9722	.9727	0	1	1	2	2	3	3	4	4
9.4	.9731	.9736	.9741	.9745	.9750	.9754	.9759	.9763	.9768	.9773	0	1	1	2	2	3	3	4	4
9.5	.9777	.9782	.9786	.9791	.9795	.9800	.9805	.9809	.9814	.9818	0	1	1	2	2	3	3	4	4
9.6	.9823	.9827	.9832	.9836	.9841	.9845	.9850	.9854	.9859	.9863	0	1	1	2	2	3	3	4	4
9.7	.9868	.9872	.9877	.9881	.9886	.9890	.9894	.9899	.9903	.9908	0	1	1	2	2	3	3	4	4
9.8	.9912	.9917	.9921	.9926	.9930	.9934	.9939	.9943	.9948	.9952	0	1	1	2	2	3	3	4	4
9.9	.9956	.9961	.9965	.9969	.9974	.9978	.9983	.9987	.9991	.9996	0	1	1	2	2	3	3	3	4

In each of the above logarithms the characteristic is 0.

SIX-PLACE LOGARITHMS. 100.0–149.9

N	0	1	2	3	4	5	6	7	8	9
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174
102	8600	9026	9451	9876	*300	*724	1147	1570	1993	2415
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616
104	7033	7451	7868	8284	8700	9116	9532	9947	*361	*775
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
107	9384	9789	*195	*600	1004	1408	1812	2216	2619	3021
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028
109	7426	7825	8223	8620	9017	9414	9811	*207	*602	*998
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830
112	9218	9606	9993	*380	*766	1153	1538	1924	2309	2694
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	*320
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
117	8186	8557	8928	9298	9668	**38	*407	*776	1145	1514
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
120	079181	9543	9904	*266	*626	*987	1347	1707	2067	2426
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552
123	9905	*258	*611	*963	1315	1667	2018	2370	2721	3071
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	**26
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	*253
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609
130	113943	4277	4611	4944	5278	5611	5943	6276	6608	6940
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	*245
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	**12
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564
138	9879	*194	*508	*822	1136	1450	1763	2076	2389	2702
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818
140	146128	6438	6748	7058	7367	7676	7985	8294	8603	8911
141	9219	9527	9835	*142	*449	*756	1063	1370	1676	1982
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
144	8362	8664	8965	9266	9567	9868	*168	*469	*769	1068
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802

SIX-PLACE LOGARITHMS. 150.0–199.9

N	0	1	2	3	4	5	6	7	8	9
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689
151	8977	9264	9552	9839	*126	*413	*699	*985	1272	1558
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	**51
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382
158	8657	8932	9206	9481	9755	**29	*303	*577	*850	1124
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848
160	204120	4391	4663	4934	5204	5475	5746	6016	6286	6556
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247
162	9515	9783	**51	*319	*586	*853	1121	1388	1654	1921
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	*193
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795
173	8046	8297	8548	8799	9049	9299	9550	9800	**50	*300
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	*176
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031
180	255273	5514	5755	5996	6237	6477	6718	6958	7198	7439
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279
186	9513	9746	9980	*213	*446	*679	*912	1144	1377	1609
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525
190	278754	8982	9211	9439	9667	9895	*123	*351	*578	*806
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635
199	8853	9071	9289	9507	9725	9943	*161	*378	*595	*813

SIX-PLACE LOGARITHMS. 200.0–249.9

N	0	1	2	3	4	5	6	7	8	9
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417
204	9630	9843	**56	*268	*481	*693	*906	1118	1330	1542
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012
210	322219	2426	2633	2839	3046	3252	3458	3665	3871	4077
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176
213	8380	8583	8787	8991	9194	9398	9601	9805	***8	*211
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257
218	8456	8656	8855	9054	9253	9451	9650	9849	**47	*246
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225
220	342423	2620	2817	3014	3212	3409	3606	3802	3999	4196
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	**54
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646
229	9835	**25	*215	*404	*593	*783	*972	1161	1350	1539
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030
234	9216	9401	9587	9772	9958	*143	*328	*513	*698	*883
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	**30
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989
245	9166	9343	9520	9698	9875	**51	*228	*405	*582	*759
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766

SIX-PLACE LOGARITHMS. 250.0–299.9

N	0	1	2	3	4	5	6	7	8	9
250	397940	8114	8287	8461	8634	8808	8981	9154	9328	9501
251	9674	9847	**20	*192	*365	*538	*711	*883	1056	1228
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764
257	9933	*102	*271	*440	*609	*777	*946	1114	1283	1451
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806
260	414973	5140	5307	5474	5641	5808	5974	6141	6308	6474
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791
263	9956	*121	*286	*451	*616	*781	*945	1110	1275	1439
264	421604	1788	1933	2097	2261	2426	2590	2754	2918	3082
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591
269	9752	9914	**75	*236	*398	*559	*720	*881	1042	1203
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004
273	6163	6322	6481	6640	6798	6957	7116	7275	7433	7592
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175
275	9333	9491	9648	9806	9964	*122	*279	*437	*594	*752
276	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003
280	447158	7313	7468	7623	7778	7933	8088	8242	8397	8552
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	**95
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242
288	9392	9543	9694	9845	9995	*146	*296	*447	*597	*748
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248
290	462398	2548	2697	2847	2997	3146	3296	3445	3594	3744
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675
295	9822	9969	*116	*263	*410	*557	*704	*851	*998	1145
296	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976

SIX-PLACE LOGARITHMS. 300.0–349.9

N	0	1	2	3	4	5	6	7	8	9
300	477121	7266	7411	7555	7700	7844	7989	8133	8278	8422
301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863
302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818
309	9958	**99	*239	*380	*520	*661	*801	*941	1081	1222
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550
316	9687	9824	9962	**99	*236	*374	*511	*648	*785	*922
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014
320	505150	5286	5421	5557	5693	5828	5964	6099	6234	6370
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068
323	9203	9337	9471	9606	9740	9874	***9	*143	*277	*411
324	510545	0679	0813	0947	1081	1215	1349	1482	1616	1750
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4414
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382
330	518514	8646	8777	8909	9040	9171	9303	9434	9566	9697
331	9828	9959	**90	*221	*353	*484	*615	*745	*876	1007
332	521138	1269	1400	1530	1661	1792	1922	2053	2183	2314
333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501
337	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788
338	8917	9045	9174	9302	9430	9559	9687	9815	9943	**72
339	530200	0328	0456	0584	0712	0840	0968	1096	1223	1351
340	531479	1607	1734	1862	1990	2117	2245	2372	2500	2627
341	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899
342	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167
343	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432
344	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693
345	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951
346	9076	9202	9327	9452	9578	9703	9829	9954	**79	*204
347	540329	0455	0580	0705	0830	0955	1080	1205	1330	1454
348	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701
349	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944

SIX-PLACE LOGARITHMS. 350.0–399.9

N	0	1	2	3	4	5	6	7	8	9
350	544068	4192	4316	4440	4564	4688	4812	4936	5060	5183
351	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419
352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881
354	9003	9126	9249	9371	9494	9616	9739	9861	9984	*106
355	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328
356	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547
357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973
359	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182
360	556303	6423	6544	6664	6785	6905	7026	7146	7267	7387
361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589
362	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787
363	9907	**26	*146	*265	*385	*504	*624	*743	*863	*982
364	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362
366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084
370	568202	8319	8436	8554	8671	8788	8905	9023	9140	9257
371	9374	9491	9608	9725	9842	9959	**76	*193	*309	*426
372	570543	0660	0776	0893	1010	1126	1243	1359	1476	1592
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072
376	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525
379	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669
380	579784	9898	**12	*126	*241	*355	*469	*583	*697	*811
381	580925	1039	1153	1267	1381	1495	1608	1722	1836	1950
382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720
388	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838
389	9950	**61	*173	*284	*396	*507	*619	*730	*842	*953
390	591065	1176	1287	1399	1510	1621	1732	1843	1955	2066
391	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175
392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681
397	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774
398	9883	9992	*101	*210	*319	*428	*537	*646	*755	*864
399	600973	1082	1191	1299	1408	1517	1625	1734	1843	1951

SIX-PLACE LOGARITHMS. 400.0–449.9

N	0	1	2	3	4	5	6	7	8	9
400	602060	2169	2277	2386	2494	2603	2711	2819	2928	3036
401	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488
407	9594	9701	9808	9914	**21	*128	*234	*341	*447	*554
408	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678
410	612784	2890	2996	3102	3207	3313	3419	3525	3630	3736
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845
413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	**32
417	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146
420	623249	3353	3456	3559	3663	3766	3869	3973	4076	4179
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308
426	9410	9512	9613	9715	9817	9919	**21	*123	*224	*326
427	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356
429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367
430	633468	3569	3670	3771	3872	3973	4074	4175	4276	4376
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389
435	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387
436	9486	9586	9686	9785	9885	9984	**84	*183	*283	*382
437	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366
439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354
440	643453	3551	3650	3749	3847	3946	4044	4143	4242	4340
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237
446	9335	9432	9530	9627	9724	9821	9919	**16	*113	*210
447	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116

SIX-PLACE LOGARITHMS. 450.0–499.9

N	0	1	2	3	4	5	6	7	8	9
450	653213	3309	3405	3502	3598	3695	3791	3888	3984	4080
451	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042
452	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870
456	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821
457	9916	**11	*106	*201	*296	*391	*486	*581	*676	*771
458	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718
459	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663
460	662758	2852	2947	3041	3135	3230	3324	3418	3512	3607
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293
466	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224
467	9317	9410	9503	9596	9689	9782	9875	9967	**60	*153
468	670246	0339	0431	0524	0617	0710	0802	0895	0988	1080
469	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005
470	672098	2190	2283	2375	2467	2560	2652	2744	2836	2929
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850
472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427
477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337
478	9428	9519	9610	9700	9791	9882	9973	**63	*154	*245
479	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151
480	681241	1332	1422	1513	1603	1693	1784	1874	1964	2055
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220
489	9309	9398	9486	9575	9664	9753	9841	9930	**19	*107
490	690196	0285	0373	0462	0550	0639	0728	0816	0905	0993
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883

SIX-PLACE LOGARITHMS. 500.0–549.9

N	0	1	2	3	4	5	6	7	8	9
500	698970	9057	9144	9231	9317	9404	9491	9578	9664	9751
501	9838	9924	**11	**98	*184	*271	*358	*444	*531	*617
502	700704	0790	0877	0963	1050	1136	1222	1309	1395	1482
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485
510	707570	7655	7740	7826	7911	7996	8081	8166	8251	8336
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	**33
513	710117	0202	0287	0371	0456	0540	0625	0710	0794	0879
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566
516	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920
520	716003	6087	6170	6254	6337	6421	6504	6588	6671	6754
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	**77
525	720159	0242	0325	0407	0490	0573	0655	0738	0821	0903
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194
530	724276	4358	4440	4522	4604	4685	4767	4849	4931	5013
531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893
537	9974	**55	*136	*217	*298	*378	*459	*540	*621	*702
538	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313
540	732394	2474	2555	2635	2715	2796	2876	2956	3037	3117
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317
545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493
549	9572	9651	9731	9810	9889	9968	**47	*126	*205	*284

SIX-PLACE LOGARITHMS. 550.0–599.9

N	0	1	2	3	4	5	6	7	8	9
550	740363	0442	0521	0600	0678	0757	0836	0915	0994	1073
551	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860
552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110
560	748188	8266	8343	8421	8498	8576	8653	8731	8808	8885
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659
562	9736	9814	9891	9968	**45	*123	*200	*277	*354	*431
563	750508	0586	0663	0740	0817	0894	0971	1048	1125	1202
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799
570	755875	5951	6027	6103	6180	6256	6332	6408	6484	6560
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836
574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592
575	9668	9743	9819	9894	9970	**45	*121	*196	*272	*347
576	760422	0498	0573	0649	0724	0799	0875	0950	1025	1101
577	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853
578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604
579	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353
580	763428	3503	3578	3653	3727	3802	3877	3952	4027	4101
581	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848
582	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594
583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303
588	9377	9451	9525	9599	9673	9746	9820	9894	9968	**42
589	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778
590	770852	0926	0999	1073	1146	1220	1293	1367	1440	1514
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354
599	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079

SIX-PLACE LOGARITHMS. 600.0–649.9

N	0	1	2	3	4	5	6	7	8	9
600	778151	8224	8296	8368	8441	8513	8585	8658	8730	8802
601	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524
602	9596	9669	9741	9813	9885	9957	**29	*101	*173	*245
603	780317	0389	0461	0533	0605	0677	0749	0821	0893	0965
604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259
610	785330	5401	5472	5543	5615	5686	5757	5828	5899	5970
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510
616	9581	9651	9722	9792	9863	9933	***4	**74	*144	*215
617	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322
620	792392	2462	2532	2602	2672	2742	2812	2882	2952	3022
621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115
624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272
630	799341	9409	9478	9547	9616	9685	9754	9823	9892	9961
631	800029	0098	0167	0236	0305	0373	0442	0511	0580	0648
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112
640	806180	6248	6316	6384	6451	6519	6587	6655	6723	6790
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492
645	9560	9627	9694	9762	9829	9896	9964	**31	**98	*165
646	810233	0300	0367	0434	0501	0569	0636	0703	0770	0837
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847

SIX-PLACE LOGARITHMS. 650.0–699.9

N	0	1	2	3	4	5	6	7	8	9
650	812913	2980	3047	3114	3181	3247	3314	3381	3448	3514
651	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181
652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175
655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838
656	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499
657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820
659	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478
660	819544	9610	9676	9741	9807	9873	9939	***4	**70	*136
661	820201	0267	0333	0399	0464	0530	0595	0661	0727	0792
662	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448
663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103
664	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010
670	826075	6140	6204	6269	6334	6399	6464	6528	6593	6658
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951
673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239
675	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882
676	9947	**11	**75	*139	*204	*268	*332	*396	*460	*525
677	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166
678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445
680	832509	2573	2637	2700	2764	2828	2892	2956	3020	3083
681	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721
682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786
690	838849	8912	8975	9038	9101	9164	9227	9289	9352	9415
691	9478	9541	9604	9667	9729	9792	9855	9918	9981	**43
692	840106	0169	0232	0294	0357	0420	0482	0545	0608	0671
693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297
694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793
698	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415
699	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036

SIX-PLACE LOGARITHMS. 700.0–749.9

N	0	1	2	3	4	5	6	7	8	9
700	845098	5160	5222	5284	5346	5408	5470	5532	5594	5656
701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275
702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972
708	850033	0095	0156	0217	0279	0340	0401	0462	0524	0585
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197
710	851258	1320	1381	1442	1503	1564	1625	1686	1747	1809
711	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272
720	857332	7393	7453	7513	7574	7634	7694	7755	7815	7875
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679
724	9739	9799	9859	9918	9978	**38	**98	*158	*218	*278
725	860338	0398	0458	0518	0578	0637	0697	0757	0817	0877
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263
730	863323	3382	3442	3501	3561	3620	3680	3739	3799	3858
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586
739	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173
740	869232	9290	9349	9408	9466	9525	9584	9642	9701	9760
741	9818	9877	9935	9994	**53	*111	*170	*228	*287	*345
742	870404	0462	0521	0579	0638	0696	0755	0813	0872	0930
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262
747	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844
748	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424
749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003

SIX-PLACE LOGARITHMS. 750.0–799.9

N	0	1	2	3	4	5	6	7	8	9
750	875061	5119	5177	5235	5293	5351	5409	5466	5524	5582
751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160
752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612
758	9669	9726	9784	9841	9898	9956	**13	**70	*127	*185
759	880242	0299	0356	0413	0471	0528	0585	0642	0699	0756
760	880814	0871	0928	0985	1042	1099	1156	1213	1271	1328
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605
765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739
767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434
770	886491	6547	6604	6660	6716	6773	6829	6885	6942	6998
771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806
776	9862	9918	9974	**30	**86	*141	*197	*253	*309	*365
777	890421	0477	0533	0589	0645	0700	0756	0812	0868	0924
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039
780	892095	2150	2206	2262	2317	2373	2429	2484	2540	2595
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572
790	897627	7682	7737	7792	7847	7902	7957	8012	8067	8122
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766
794	9821	9875	9930	9985	**39	**94	*149	*203	*258	*312
795	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036

SIX-PLACE LOGARITHMS. 800.0–849.9

N	0	1	2	3	4	5	6	7	8	9
800	903090	3144	3199	3253	3307	3361	3416	3470	3524	3578
801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431
810	908485	8539	8592	8646	8699	8753	8807	8860	8914	8967
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	·9503
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	**37
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761
820	913814	3867	3920	3973	4026	4079	4132	4184	4237	4290
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026
830	919078	9130	9183	9235	9287	9340	9392	9444	9496	9549
831	9601	9653	9706	9758	9810	9862	9914	9967	**19	**71
832	920123	0176	0228	0280	0332	0384	0436	0489	0541	0593
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228
840	924279	4331	4383	4434	4486	4538	4589	4641	4693	4744
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832
847	7883	7935	7986	8037	8088	8140	9191	8242	8293	8345
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368

SIX-PLACE LOGARITHMS. 850.0–899.9

N	0	1	2	3	4	5	6	7	8	9
850	929419	9470	9521	9572	9623	9674	9725	9776	9827	9879
851	9930	9981	**32	**83	*134	*185	*236	*287	*338	*389
852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448
860	934498	4549	4599	4650	4700	4751	4801	4852	4902	4953
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9169
870	939519	9569	9619	9669	9719	9769	9819	9869	9918	9968
871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433
880	944483	4532	4581	4631	4680	4729	4779	4828	4877	4927
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341
890	949390	9439	9488	9536	9585	9634	9683	9731	9780	9829
891	9878	9926	9975	**24	**73	*121	*170	*219	*267	*316
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194

SIX-PLACE LOGARITHMS. 900.0–949.9

N	0	1	2	3	4	5	6	7	8	9
900	954243	4291	4339	4387	4435	4484	4532	4580	4628	4677
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994
910	959041	9089	9137	9185	9232	9280	9328	9375	9423	9471
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947
912	9995	**42	**90	*138	*185	*233	*280	*328	*376	*423
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741
920	963788	3835	3882	3929	3977	4024	4071	4118	4165	4212
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436
930	968483	8530	8576	8623	8670	8716	8763	8810	8856	8903
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835
933	9882	9928	9975	**21	**68	*114	*161	*207	*254	*300
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082
940	973128	3174	3220	3266	3313	3359	3405	3451	3497	3543
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678

SIX-PLACE LOGARITHMS. 950.0–999.9

N	0	1	2	3	4	5	6	7	8	9
950	977724	7769	7815	7861	7906	7952	7998	8043	8089	8135
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226
960	982271	2316	2362	2407	2452	2497	2543	2588	2633	2678
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727
970	986772	6817	6861	6906	6951	6996	7040	7085	7130	7175
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850
977	9895	9939	9983	**28	**72	*117	*161	*206	*250	*294
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182
980	991226	1270	1315	1359	1403	1448	1492	1536	1580	1625
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591
990	995635	5679	5723	5767	5811	5854	5898	5942	5986	6030
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957

4. Trigonometric Functions

Natural Functions. On pages 336–341 are given four-place tables of natural sines, cosines, and tangents, — the three most important functions of an angle. In these tables the functions are given for each 0.1°, or 6′, from 0° to 90°. By means of the columns of differences, which are given at the right of each page, or by using ordinary interpolation, the tables are easily extended to permit finding the functions of an angle measured to the nearest 1′.

In cases requiring greater accuracy, tables with a larger number of decimal places are used, but for ordinary work, such as is emphasized in this book, the tables given here are sufficiently accurate.

The methods of using the tables of natural functions are already familiar to the student, not only in finding the function of a given angle, but also in finding the angle which has a given function, as described on page 214.

In finding the tangent of an angle greater than 75° it is necessary to use ordinary interpolation, which is described on page 214.

In the table of tangents the heavy-face type indicates that the integral part of the tangent, which is given in the column headed 0.0°, or 0′, is to be increased by 1 at this point. For example, on page 341 the tangent of 71° 36′ is printed in heavy-face type, and hence tan 71° 36′ = 3.0061, tan 71° 42′ = 3.0237, and so on for the rest of the line.

In using the tables to find the natural functions of an obtuse angle, the formulas

$$\sin(180° - \theta) = \sin\theta,$$

$$\cos(180° - \theta) = -\cos\theta,$$

and

$$\tan(180° - \theta) = -\tan\theta,$$

which were given on page 221, are first used in order to reduce the functions of an obtuse angle to those of an acute angle.

For example, in finding the three functions of 146° we have

$$\sin 146° = \sin(180° - 146°) = \sin 34° = 0.5592,$$

$$\cos 146° = -\cos(180° - 146°) = -\cos 34° = -0.8290,$$

$$\tan 146° = -\tan(180° - 146°) = -\tan 34° = -0.6745.$$

Logarithmic Functions. On pages 342–347 are given four-place tables of the logarithms of the sines, cosines, and tangents of angles from 0° to 90° for each 0.1°, or 6′. As in the case of the tables of natural functions, by using the difference columns, or the method of ordinary interpolation, the tables can be extended to permit finding the logarithmic functions of an angle measured to the nearest 1′.

As with the natural functions, differences for the sine and the tangent are to be added, while those for the cosine are to be subtracted.

The methods of using the tables are a combination, as described on page 215, of the methods used in connection with the table of four-place logarithms and those used with the tables of natural functions.

In the tables it should be observed that the characteristics are given in the column headed **0.0°**, or **0′**, and that, unless changed by the heavy-face type, the characteristic is the same for the entire line. The heavy-face type indicates a decrease or an increase (according to the table used) in the numerical value of the characteristic at that point.

In many tables the characteristics of the logarithmic functions are given 10 too large on account of the difficulty of printing negative characteristics; that is, log tan 55° 30′ is given as 10.1629, and it is understood that 10 is to be subtracted. In the tables in this book a special type has been devised which makes it unnecessary to resort to such a plan. Attention should be called to this point in case students are required to use other tables in examinations.

The logarithmic functions of obtuse angles can be found from the tables by applying the reduction formulas on page 334 in the same way as for the natural functions.

For example, in finding the logarithmic functions of 146° we have

$$\log \sin 146° = \log \sin (180° - 146°) = \log \sin 34° = \bar{1}.7476,$$
$$\log \cos 146° = -\log \cos (180° - 146°) = -\log \cos 34° = -\bar{1}.9186,$$
$$\log \tan 146° = -\log \tan (180° - 146°) = -\log \tan 34° = -\bar{1}.8290.$$

In using the logarithmic cosines and tangents of obtuse angles, particular attention should be paid to the negative signs.

NATURAL SINES. 0°–45°

○	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	.0000	.0017	.0035	.0052	.0070	.0087	.0105	.0122	.0140	.0157	3	6	9	12	15
1	.0175	.0192	.0209	.0227	.0244	.0262	.0279	.0297	.0314	.0332	3	6	9	12	15
2	.0349	.0366	.0384	.0401	.0419	.0436	.0454	.0471	.0488	.0506	3	6	9	12	15
3	.0523	.0541	.0558	.0576	.0593	.0610	.0628	.0645	.0663	.0680	3	6	9	12	15
4	.0698	.0715	.0732	.0750	.0767	.0785	.0802	.0819	.0837	.0854	3	6	9	12	14
5	.0872	.0889	.0906	.0924	.0941	.0958	.0976	.0993	.1011	.1028	3	6	9	12	14
6	.1045	.1063	.1080	.1097	.1115	.1132	.1149	.1167	.1184	.1201	3	6	9	12	14
7	.1219	.1236	.1253	.1271	.1288	.1305	.1323	.1340	.1357	.1374	3	6	9	12	14
8	.1392	.1409	.1426	.1444	.1461	.1478	.1495	.1513	.1530	.1547	3	6	9	12	14
9	.1564	.1582	.1599	.1616	.1633	.1650	.1668	.1685	.1702	.1719	3	6	9	12	14
10	.1736	.1754	.1771	.1788	.1805	.1822	.1840	.1857	.1874	.1891	3	6	9	11	14
11	.1908	.1925	.1942	.1959	.1977	.1994	.2011	.2028	.2045	.2062	3	6	9	11	14
12	.2079	.2096	.2113	.2130	.2147	.2164	.2181	.2198	.2215	.2233	3	6	9	11	14
13	.2250	.2267	.2284	.2300	.2317	.2334	.2351	.2368	.2385	.2402	3	6	8	11	14
14	.2419	.2436	.2453	.2470	.2487	.2504	.2521	.2538	.2554	.2571	3	6	8	11	14
15	.2588	.2605	.2622	.2639	.2656	.2672	.2689	.2706	.2723	.2740	3	6	8	11	14
16	.2756	.2773	.2790	.2807	.2823	.2840	.2857	.2874	.2890	.2907	3	6	8	11	14
17	.2924	.2940	.2957	.2974	.2990	.3007	.3024	.3040	.3057	.3074	3	6	8	11	14
18	.3090	.3107	.3123	.3140	.3156	.3173	.3190	.3206	.3223	.3239	3	6	8	11	14
19	.3256	.3272	.3289	.3305	.3322	.3338	.3355	.3371	.3387	.3404	3	5	8	11	14
20	.3420	.3437	.3453	.3469	.3486	.3502	.3518	.3535	.3551	.3567	3	5	8	11	14
21	.3584	.3600	.3616	.3633	.3649	.3665	.3681	.3697	.3714	.3730	3	5	8	11	14
22	.3746	.3762	.3778	.3795	.3811	.3827	.3843	.3859	.3875	.3891	3	5	8	11	14
23	.3907	.3923	.3939	.3955	.3971	.3987	.4003	.4019	.4035	.4051	3	5	8	11	14
24	.4067	.4083	.4099	.4115	.4131	.4147	.4163	.4179	.4195	.4210	3	5	8	11	13
25	.4226	.4242	.4258	.4274	.4289	.4305	.4321	.4337	.4352	.4368	3	5	8	11	13
26	.4384	.4399	.4415	.4431	.4446	.4462	.4478	.4493	.4509	.4524	3	5	8	10	13
27	.4540	.4555	.4571	.4586	.4602	.4617	.4633	.4648	.4664	.4679	3	5	8	10	13
28	.4695	.4710	.4726	.4741	.4756	.4772	.4787	.4802	.4818	.4833	3	5	8	10	13
29	.4848	.4863	.4879	.4894	.4909	.4924	.4939	.4955	.4970	.4985	3	5	8	10	13
30	.5000	.5015	.5030	.5045	.5060	.5075	.5090	.5105	.5120	.5135	3	5	8	10	13
31	.5150	.5165	.5180	.5195	.5210	.5225	.5240	.5255	.5270	.5284	2	5	7	10	12
32	.5299	.5314	.5329	.5344	.5358	.5373	.5388	.5402	.5417	.5432	2	5	7	10	12
33	.5446	.5461	.5476	.5490	.5505	.5519	.5534	.5548	.5563	.5577	2	5	7	10	12
34	.5592	.5606	.5621	.5635	.5650	.5664	.5678	.5693	.5707	.5721	2	5	7	10	12
35	.5736	.5750	.5764	.5779	.5793	.5807	.5821	.5835	.5850	.5864	2	5	7	9	12
36	.5878	.5892	.5906	.5920	.5934	.5948	.5962	.5976	.5990	.6004	2	5	7	9	12
37	.6018	.6032	.6046	.6060	.6074	.6088	.6101	.6115	.6129	.6143	2	5	7	9	12
38	.6157	.6170	.6184	.6198	.6211	.6225	.6239	.6252	.6266	.6280	2	5	7	9	11
39	.6293	.6307	.6320	.6334	.6347	.6361	.6374	.6388	.6401	.6414	2	4	7	9	11
40	.6428	.6441	.6455	.6468	.6481	.6494	.6508	.6521	.6534	.6547	2	4	7	9	11
41	.6561	.6574	.6587	.6600	.6613	.6626	.6639	.6652	.6665	.6678	2	4	7	9	11
42	.6691	.6704	.6717	.6730	.6743	.6756	.6769	.6782	.6794	.6807	2	4	6	9	11
43	.6820	.6833	.6845	.6858	.6871	.6884	.6896	.6909	.6921	.6934	2	4	6	8	11
44	.6947	.6959	.6972	.6984	.6997	.7009	.7022	.7034	.7046	.7059	2	4	6	8	10

All the above sines are less than 1.

NATURAL SINES. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	.7071	.7083	.7096	.7108	.7120	.7133	.7145	.7157	.7169	.7181	2	4	6	8	10
46	.7193	.7206	.7218	.7230	.7242	.7254	.7266	.7278	.7290	.7302	2	4	6	8	10
47	.7314	.7325	.7337	.7349	.7361	.7373	.7385	.7396	.7408	.7420	2	4	6	8	10
48	.7431	.7443	.7455	.7466	.7478	.7490	.7501	.7513	.7524	.7536	2	4	6	8	10
49	.7547	.7559	.7570	.7581	.7593	.7604	.7615	.7627	.7638	.7649	2	4	6	8	9
50	.7660	.7672	.7683	.7694	.7705	.7716	.7727	.7738	.7749	.7760	2	4	6	7	9
51	.7771	.7782	.7793	.7804	.7815	.7826	.7837	.7848	.7859	.7869	2	4	5	7	9
52	.7880	.7891	.7902	.7912	.7923	.7934	.7944	.7955	.7965	.7976	2	4	5	7	9
53	.7986	.7997	.8007	.8018	.8028	.8039	.8049	.8059	.8070	.8080	2	3	5	7	9
54	.8090	.8100	.8111	.8121	.8131	.8141	.8151	.8161	.8171	.8181	2	3	5	7	8
55	.8192	.8202	.8211	.8221	.8231	.8241	.8251	.8261	.8271	.8281	2	3	5	7	8
56	.8290	.8300	.8310	.8320	.8329	.8339	.8348	.8358	.8368	.8377	2	3	5	6	8
57	.8387	.8396	.8406	.8415	.8425	.8434	.8443	.8453	.8462	.8471	2	3	5	6	8
58	.8480	.8490	.8499	.8508	.8517	.8526	.8536	.8545	.8554	.8563	2	3	5	6	8
59	.8572	.8581	.8590	.8599	.8607	.8616	.8625	.8634	.8643	.8652	1	3	4	6	7
60	.8660	.8669	.8678	.8686	.8695	.8704	.8712	.8721	.8729	.8738	1	3	4	6	7
61	.8746	.8755	.8763	.8771	.8780	.8788	.8796	.8805	.8813	.8821	1	3	4	6	7
62	.8829	.8838	.8846	.8854	.8862	.8870	.8878	.8886	.8894	.8902	1	3	4	5	7
63	.8910	.8918	.8926	.8934	.8942	.8949	.8957	.8965	.8973	.8980	1	3	4	5	6
64	.8988	.8996	.9003	.9011	.9018	.9026	.9033	.9041	.9048	.9056	1	3	4	5	6
65	.9063	.9070	.9078	.9085	.9092	.9100	.9107	.9114	.9121	.9128	1	2	4	5	6
66	.9135	.9143	.9150	.9157	.9164	.9171	.9178	.9184	.9191	.9198	1	2	3	5	6
67	.9205	.9212	.9219	.9225	.9232	.9239	.9245	.9252	.9259	.9265	1	2	3	4	6
68	.9272	.9278	.9285	.9291	.9298	.9304	.9311	.9317	.9323	.9330	1	2	3	4	5
69	.9336	.9342	.9348	.9354	.9361	.9367	.9373	.9379	.9385	.9391	1	2	3	4	5
70	.9397	.9403	.9409	.9415	.9421	.9426	.9432	.9438	.9444	.9449	1	2	3	4	5
71	.9455	.9461	.9466	.9472	.9478	.9483	.9489	.9494	.9500	.9505	1	2	3	4	5
72	.9511	.9516	.9521	.9527	.9532	.9537	.9542	.9548	.9553	.9558	1	2	3	4	4
73	.9563	.9568	.9573	.9578	.9583	.9588	.9593	.9598	.9603	.9608	1	2	2	3	4
74	.9613	.9617	.9622	.9627	.9632	.9636	.9641	.9646	.9650	.9655	1	2	2	3	4
75	.9659	.9664	.9668	.9673	.9677	.9681	.9686	.9690	.9694	.9699	1	1	2	3	4
76	.9703	.9707	.9711	.9715	.9720	.9724	.9728	.9732	.9736	.9740	1	1	2	3	3
77	.9744	.9748	.9751	.9755	.9759	.9763	.9767	.9770	.9774	.9778	1	1	2	3	3
78	.9781	.9785	.9789	.9792	.9796	.9799	.9803	.9806	.9810	.9813	1	1	2	2	3
79	.9816	.9820	.9823	.9826	.9829	.9833	.9836	.9839	.9842	.9845	1	1	2	2	3
80	.9848	.9851	.9854	.9857	.9860	.9863	.9866	.9869	.9871	.9874	0	1	1	2	2
81	.9877	.9880	.9882	.9885	.9888	.9890	.9893	.9895	.9898	.9900	0	1	1	2	2
82	.9903	.9905	.9907	.9910	.9912	.9914	.9917	.9919	.9921	.9923	0	1	1	2	2
83	.9925	.9928	.9930	.9932	.9934	.9936	.9938	.9940	.9942	.9943	0	1	1	1	2
84	.9945	.9947	.9949	.9951	.9952	.9954	.9956	.9957	.9959	.9960	0	1	1	1	1
85	.9962	.9963	.9965	.9966	.9968	.9969	.9971	.9972	.9973	.9974	0	0	1	1	1
86	.9976	.9977	.9978	.9979	.9980	.9981	.9982	.9983	.9984	.9985	0	0	1	1	1
87	.9986	.9987	.9988	.9989	.9990	.9990	.9991	.9992	.9993	.9993	0	0	0	1	1
88	.9994	.9995	.9995	.9996	.9996	.9997	.9997	.9997	.9998	.9998	0	0	0	0	0
89	.9998	.9999	.9999	.9999	.9999	1.000	1.000	1.000	1.000	1.000	0	0	0	0	0

The precise value of all sines except sin 90° is less than 1.

NATURAL COSINES. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	— Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	1.000	1.000	1.000	1.000	1.000	1.000	.9999	.9999	.9999	.9999	0	0	0	0	0
1	.9998	.9998	.9998	.9997	.9997	.9997	.9996	.9996	.9995	.9995	0	0	0	0	0
2	.9994	.9993	.9993	.9992	.9991	.9990	.9990	.9989	.9988	.9987	0	0	0	0	0
3	.9986	.9985	.9984	.9983	.9982	.9981	.9980	.9979	.9978	.9977	0	0	1	1	1
4	.9976	.9974	.9973	.9972	.9971	.9969	.9968	.9966	.9965	.9963	0	0	1	1	1
5	.9962	.9960	.9959	.9957	.9956	.9954	.9952	.9951	.9949	.9947	0	1	1	1	1
6	.9945	.9943	.9942	.9940	.9938	.9936	.9934	.9932	.9930	.9928	0	1	1	1	2
7	.9925	.9923	.9921	.9919	.9917	.9914	.9912	.9910	.9907	.9905	0	1	1	2	2
8	.9903	.9900	.9898	.9895	.9893	.9890	.9888	.9885	.9882	.9880	0	1	1	2	2
9	.9877	.9874	.9871	.9869	.9866	.9863	.9860	.9857	.9854	.9851	0	1	1	2	2
10	.9848	.9845	.9842	.9839	.9836	.9833	.9829	.9826	.9823	.9820	1	1	2	2	3
11	.9816	.9813	.9810	.9806	.9803	.9799	.9796	.9792	.9789	.9785	1	1	2	2	3
12	.9781	.9778	.9774	.9770	.9767	.9763	.9759	.9755	.9751	.9748	1	1	2	3	3
13	.9744	.9740	.9736	.9732	.9728	.9724	.9720	.9715	.9711	.9707	1	1	2	3	3
14	.9703	.9699	.9694	.9690	.9686	.9681	.9677	.9673	.9668	.9664	1	1	2	3	4
15	.9659	.9655	.9650	.9646	.9641	.9636	.9632	.9627	.9622	.9617	1	2	2	3	4
16	.9613	.9608	.9603	.9598	.9593	.9588	.9583	.9578	.9573	.9568	1	2	2	3	4
17	.9563	.9558	.9553	.9548	.9542	.9537	.9532	.9527	.9521	.9516	1	2	3	4	4
18	.9511	.9505	.9500	.9494	.9489	.9483	.9478	.9472	.9466	.9461	1	2	3	4	5
19	.9455	.9449	.9444	.9438	.9432	.9426	.9421	.9415	.9409	.9403	1	2	3	4	5
20	.9397	.9391	.9385	.9379	.9373	.9367	.9361	.9354	.9348	.9342	1	2	3	4	5
21	.9336	.9330	.9323	.9317	.9311	.9304	.9298	.9291	.9285	.9278	1	2	3	4	5
22	.9272	.9265	.9259	.9252	.9245	.9239	.9232	.9225	.9219	.9212	1	2	3	4	6
23	.9205	.9198	.9191	.9184	.9178	.9171	.9164	.9157	.9150	.9143	1	2	3	5	6
24	.9135	.9128	.9121	.9114	.9107	.9100	.9092	.9085	.9078	.9070	1	2	4	5	6
25	.9063	.9056	.9048	.9041	.9033	.9026	.9018	.9011	.9003	.8996	1	3	4	5	6
26	.8988	.8980	.8973	.8965	.8957	.8949	.8942	.8934	.8926	.8918	1	3	4	5	6
27	.8910	.8902	.8894	.8886	.8878	.8870	.8862	.8854	.8846	.8838	1	3	4	5	7
28	.8829	.8821	.8813	.8805	.8796	.8788	.8780	.8771	.8763	.8755	1	3	4	6	7
29	.8746	.8738	.8729	.8721	.8712	.8704	.8695	.8686	.8678	.8669	1	3	4	6	7
30	.8660	.8652	.8643	.8634	.8625	.8616	.8607	.8599	.8590	.8581	1	3	4	6	7
31	.8572	.8563	.8554	.8545	.8536	.8526	.8517	.8508	.8499	.8490	2	3	5	6	8
32	.8480	.8471	.8462	.8453	.8443	.8434	.8425	.8415	.8406	.8396	2	3	5	6	8
33	.8387	.8377	.8368	.8358	.8348	.8339	.8329	.8320	.8310	.8300	2	3	5	6	8
34	.8290	.8281	.8271	.8261	.8251	.8241	.8231	.8221	.8211	.8202	2	3	5	7	8
35	.8192	.8181	.8171	.8161	.8151	.8141	.8131	.8121	.8111	.8100	2	3	5	7	8
36	.8090	.8080	.8070	.8059	.8049	.8039	.8028	.8018	.8007	.7997	2	3	5	7	9
37	.7986	.7976	.7965	.7955	.7944	.7934	.7923	.7912	.7902	.7891	2	4	5	7	9
38	.7880	.7869	.7859	.7848	.7837	.7826	.7815	.7804	.7793	.7782	2	4	5	7	9
39	.7771	.7760	.7749	.7738	.7727	.7716	.7705	.7694	.7683	.7672	2	4	6	7	9
40	.7660	.7649	.7638	.7627	.7615	.7604	.7593	.7581	.7570	.7559	2	4	6	8	9
41	.7547	.7536	.7524	.7513	.7501	.7490	.7478	.7466	.7455	.7443	2	4	6	8	10
42	.7431	.7420	.7408	.7396	.7385	.7373	.7361	.7349	.7337	.7325	2	4	6	8	10
43	.7314	.7302	.7290	.7278	.7266	.7254	.7242	.7230	.7218	.7206	2	4	6	8	10
44	.7193	.7181	.7169	.7157	.7145	.7133	.7120	.7108	.7096	.7083	2	4	6	8	10

The precise value of all cosines except cos 0° is less than 1.

NATURAL COSINES. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	— Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	.7071	.7059	.7046	.7034	.7022	.7009	.6997	.6984	.6972	.6959	2	4	6	8	10
46	.6947	.6934	.6921	.6909	.6896	.6884	.6871	.6858	.6845	.6833	2	4	6	8	11
47	.6820	.6807	.6794	.6782	.6769	.6756	.6743	.6730	.6717	.6704	2	4	6	9	11
48	.6691	.6678	.6665	.6652	.6639	.6626	.6613	.6600	.6587	.6574	2	4	7	9	11
49	.6561	.6547	.6534	.6521	.6508	.6494	.6481	.6468	.6455	.6441	2	4	7	9	11
50	.6428	.6414	.6401	.6388	.6374	.6361	.6347	.6334	.6320	.6307	2	4	7	9	11
51	.6293	.6280	.6266	.6252	.6239	.6225	.6211	.6198	.6184	.6170	2	5	7	9	11
52	.6157	.6143	.6129	.6115	.6101	.6088	.6074	.6060	.6046	.6032	2	5	7	9	12
53	.6018	.6004	.5990	.5976	.5962	.5948	.5934	.5920	.5906	.5892	2	5	7	9	12
54	.5878	.5864	.5850	.5835	.5821	.5807	.5793	.5779	.5764	.5750	2	5	7	9	12
55	.5736	.5721	.5707	.5693	.5678	.5664	.5650	.5635	.5621	.5606	2	5	7	10	12
56	.5592	.5577	.5563	.5548	.5534	.5519	.5505	.5490	.5476	.5461	2	5	7	10	12
57	.5446	.5432	.5417	.5402	.5388	.5373	.5358	.5344	.5329	.5314	2	5	7	10	12
58	.5299	.5284	.5270	.5255	.5240	.5225	.5210	.5195	.5180	.5165	2	5	7	10	12
59	.5150	.5135	.5120	.5105	.5090	.5075	.5060	.5045	.5030	.5015	3	5	8	10	13
60	.5000	.4985	.4970	.4955	.4939	.4924	.4909	.4894	.4879	.4863	3	5	8	10	13
61	.4848	.4833	.4818	.4802	.4787	.4772	.4756	.4741	.4726	.4710	3	5	8	10	13
62	.4695	.4679	.4664	.4648	.4633	.4617	.4602	.4586	.4571	.4555	3	5	8	10	13
63	.4540	.4524	.4509	.4493	.4478	.4462	.4446	.4431	.4415	.4399	3	5	8	10	13
64	.4384	.4368	.4352	.4337	.4321	.4305	.4289	.4274	.4258	.4242	3	5	8	11	13
65	.4226	.4210	.4195	.4179	.4163	.4147	.4131	.4115	.4099	.4083	3	5	8	11	13
66	.4067	.4051	.4035	.4019	.4003	.3987	.3971	.3955	.3939	.3923	3	5	8	11	13
67	.3907	.3891	.3875	.3859	.3843	.3827	.3811	.3795	.3778	.3762	3	5	8	11	13
68	.3746	.3730	.3714	.3697	.3681	.3665	.3649	.3633	.3616	.3600	3	5	8	11	14
69	.3584	.3567	.3551	.3535	.3518	.3502	.3486	.3469	.3453	.3437	3	5	8	11	14
70	.3420	.3404	.3387	.3371	.3355	.3338	.3322	.3305	.3289	.3272	3	5	8	11	14
71	.3256	.3239	.3223	.3206	.3190	.3173	.3156	.3140	.3123	.3107	3	6	8	11	14
72	.3090	.3074	.3057	.3040	.3024	.3007	.2990	.2974	.2957	.2940	3	6	8	11	14
73	.2924	.2907	.2890	.2874	.2857	.2840	.2823	.2807	.2790	.2773	3	6	8	11	14
74	.2756	.2740	.2723	.2706	.2689	.2672	.2656	.2639	.2622	.2605	3	6	8	11	14
75	.2588	.2571	.2554	.2538	.2521	.2504	.2487	.2470	.2453	.2436	3	6	8	11	14
76	.2419	.2402	.2385	.2368	.2351	.2334	.2317	.2300	.2284	.2267	3	6	8	11	14
77	.2250	.2233	.2215	.2198	.2181	.2164	.2147	.2130	.2113	.2096	3	6	9	11	14
78	.2079	.2062	.2045	.2028	.2011	.1994	.1977	.1959	.1942	.1925	3	6	9	11	14
79	.1908	.1891	.1874	.1857	.1840	.1822	.1805	.1788	.1771	.1754	3	6	9	11	14
80	.1736	.1719	.1702	.1685	.1668	.1650	.1633	.1616	.1599	.1582	3	6	9	11	14
81	.1564	.1547	.1530	.1513	.1495	.1478	.1461	.1444	.1426	.1409	3	6	9	12	14
82	.1392	.1374	.1357	.1340	.1323	.1305	.1288	.1271	.1253	.1236	3	6	9	12	14
83	.1219	.1201	.1184	.1167	.1149	.1132	.1115	.1097	.1080	.1063	3	6	9	12	14
84	.1045	.1028	.1011	.0993	.0976	.0958	.0941	.0924	.0906	.0889	3	6	9	12	14
85	.0872	.0854	.0837	.0819	.0802	.0785	.0767	.0750	.0732	.0715	3	6	9	12	14
86	.0698	.0680	.0663	.0645	.0628	.0610	.0593	.0576	.0558	.0541	3	6	9	12	15
87	.0523	.0506	.0488	.0471	.0454	.0436	.0419	.0401	.0384	.0366	3	6	9	12	15
88	.0349	.0332	.0314	.0297	.0279	.0262	.0244	.0227	.0209	.0192	3	6	9	12	15
89	.0175	.0157	.0140	.0122	.0105	.0087	.0070	.0052	.0035	.0017	3	6	9	12	15

All the above cosines are less than 1.

NATURAL TANGENTS. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	0.0000	.0017	.0035	.0052	.0070	.0087	.0105	.0122	.0140	.0157	3	6	9	12	15
1	0.0175	.0192	.0209	.0227	.0244	.0262	.0279	.0297	.0314	.0332	3	6	9	12	15
2	0.0349	.0367	.0384	.0402	.0419	.0437	.0454	.0472	.0489	.0507	3	6	9	12	15
3	0.0524	.0542	.0559	.0577	.0594	.0612	.0629	.0647	.0664	.0682	3	6	9	12	15
4	0.0699	.0717	.0734	.0752	.0769	.0787	.0805	.0822	.0840	.0857	3	6	9	12	15
5	0.0875	.0892	.0910	.0928	.0945	.0963	.0981	.0998	.1016	.1033	3	6	9	12	15
6	0.1051	.1069	.1086	.1104	.1122	.1139	.1157	.1175	.1192	.1210	3	6	9	12	15
7	0.1228	.1246	.1263	.1281	.1299	.1317	.1334	.1352	.1370	.1388	3	6	9	12	15
8	0.1405	.1423	.1441	.1459	.1477	.1495	.1512	.1530	.1548	.1566	3	6	9	12	15
9	0.1584	.1602	.1620	.1638	.1655	.1673	.1691	.1709	.1727	.1745	3	6	9	12	15
10	0.1763	.1781	.1799	.1817	.1835	.1853	.1871	.1890	.1908	.1926	3	6	9	12	15
11	0.1944	.1962	.1980	.1998	.2016	.2035	.2053	.2071	.2089	.2107	3	6	9	12	15
12	0.2126	.2144	.2162	.2180	.2199	.2217	.2235	.2254	.2272	.2290	3	6	9	12	15
13	0.2309	.2327	.2345	.2364	.2382	.2401	.2419	.2438	.2456	.2475	3	6	9	12	15
14	0.2493	.2512	.2530	.2549	.2568	.2586	.2605	.2623	.2642	.2661	3	6	9	12	16
15	0.2679	.2698	.2717	.2736	.2754	.2773	.2792	.2811	.2830	.2849	3	6	9	13	16
16	0.2867	.2886	.2905	.2924	.2943	.2962	.2981	.3000	.3019	.3038	3	6	9	13	16
17	0.3057	.3076	.3096	.3115	.3134	.3153	.3172	.3191	.3211	.3230	3	6	10	13	16
18	0.3249	.3269	.3288	.3307	.3327	.3346	.3365	.3385	.3404	.3424	3	6	10	13	16
19	0.3443	.3463	.3482	.3502	.3522	.3541	.3561	.3581	.3600	.3620	3	7	10	13	16
20	0.3640	.3659	.3679	.3699	.3719	.3739	.3759	.3779	.3799	.3819	3	7	10	13	17
21	0.3839	.3859	.3879	.3899	.3919	.3939	.3959	.3979	.4000	.4020	3	7	10	13	17
22	0.4040	.4061	.4081	.4101	.4122	.4142	.4163	.4183	.4204	.4224	3	7	10	14	17
23	0.4245	.4265	.4286	.4307	.4327	.4348	.4369	.4390	.4411	.4431	3	7	10	14	17
24	0.4452	.4473	.4494	.4515	.4536	.4557	.4578	.4599	.4621	.4642	4	7	11	14	18
25	0.4663	.4684	.4706	.4727	.4748	.4770	.4791	.4813	.4834	.4856	4	7	11	14	18
26	0.4877	.4899	.4921	.4942	.4964	.4986	.5008	.5029	.5051	.5073	4	7	11	15	18
27	0.5095	.5117	.5139	.5161	.5184	.5206	.5228	.5250	.5272	.5295	4	7	11	15	18
28	0.5317	.5340	.5362	.5384	.5407	.5430	.5452	.5475	.5498	.5520	4	8	11	15	19
29	0.5543	.5566	.5589	.5612	.5635	.5658	.5681	.5704	.5727	.5750	4	8	12	15	19
30	0.5774	.5797	.5820	.5844	.5867	.5890	.5914	.5938	.5961	.5985	4	8	12	16	20
31	0.6009	.6032	.6056	.6080	.6104	.6128	.6152	.6176	.6200	.6224	4	8	12	16	20
32	0.6249	.6273	.6297	.6322	.6346	.6371	.6395	.6420	.6445	.6469	4	8	12	16	20
33	0.6494	.6519	.6544	.6569	.6594	.6619	.6644	.6669	.6694	.6720	4	8	13	17	21
34	0.6745	.6771	.6796	.6822	.6847	.6873	.6899	.6924	.6950	.6976	4	9	13	17	21
35	0.7002	.7028	.7054	.7080	.7107	.7133	.7159	.7186	.7212	.7239	4	9	13	18	22
36	0.7265	.7292	.7319	.7346	.7373	.7400	.7427	.7454	.7481	.7508	5	9	14	18	23
37	0.7536	.7563	.7590	.7618	.7646	.7673	.7701	.7729	.7757	.7785	5	9	14	18	23
38	0.7813	.7841	.7869	.7898	.7926	.7954	.7983	.8012	.8040	.8069	5	9	14	19	24
39	0.8098	.8127	.8156	.8185	.8214	.8243	.8273	.8302	.8332	.8361	5	10	15	20	24
40	0.8391	.8421	.8451	.8481	.8511	.8541	.8571	.8601	.8632	.8662	5	10	15	20	25
41	0.8693	.8724	.8754	.8785	.8816	.8847	.8878	.8910	.8941	.8972	5	10	16	21	26
42	0.9004	.9036	.9067	.9099	.9131	.9163	.9195	.9228	.9260	.9293	5	11	16	21	27
43	0.9325	.9358	.9391	.9424	.9457	.9490	.9523	.9556	.9590	.9623	6	11	17	22	28
44	0.9657	.9691	.9725	.9759	.9793	.9827	.9861	.9896	.9930	.9965	6	11	17	23	29

All tangents less than tan 45° are less than 1.

NATURAL TANGENTS. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	1.0000	.0035	.0070	.0105	.0141	.0176	.0212	.0247	.0283	.0319	6	12	18	24	30
46	1.0355	.0392	.0428	.0464	.0501	.0538	.0575	.0612	.0649	.0686	6	12	18	25	31
47	1.0724	.0761	.0799	.0837	.0875	.0913	.0951	.0990	.1028	.1067	6	13	19	25	32
48	1.1106	.1145	.1184	.1224	.1263	.1303	.1343	.1383	.1423	.1463	7	13	20	26	33
49	1.1504	.1544	.1585	.1626	.1667	.1708	.1750	.1792	.1833	.1875	7	14	21	28	34
50	1.1918	.1960	.2002	.2045	.2088	.2131	.2174	.2218	.2261	.2305	7	14	22	29	36
51	1.2349	.2393	.2437	.2482	.2527	.2572	.2617	.2662	.2708	.2753	8	15	23	30	38
52	1.2799	.2846	.2892	.2938	.2985	.3032	.3079	.3127	.3175	.3222	8	16	24	31	39
53	1.3270	.3319	.3367	.3416	.3465	.3514	.3564	.3613	.3663	.3713	8	16	25	33	41
54	1.3764	.3814	.3865	.3916	.3968	.4019	.4071	.4124	.4176	.4229	9	17	26	34	43
55	1.4281	.4335	.4388	.4442	.4496	.4550	.4605	.4659	.4715	.4770	9	18	27	36	45
56	1.4826	.4882	.4938	.4994	.5051	.5108	.5166	.5224	.5282	.5340	10	19	29	38	48
57	1.5399	.5458	.5517	.5577	.5637	.5697	.5757	.5818	.5880	.5941	10	20	30	40	50
58	1.6003	.6066	.6128	.6191	.6255	.6319	.6383	.6447	.6512	.6577	11	21	32	43	53
59	1.6643	.6709	.6775	.6842	.6909	.6977	.7045	.7113	.7182	.7251	11	23	34	45	56
60	1.7321	.7391	.7461	.7532	.7603	.7675	.7747	.7820	.7893	.7966	12	24	36	48	60
61	1.8040	.8115	.8190	.8265	.8341	.8418	.8495	.8572	.8650	.8728	13	26	38	51	64
62	1.8807	.8887	.8967	.9047	.9128	.9210	.9292	.9375	.9458	.9542	14	27	41	55	68
63	1.9626	.9711	.9797	.9883	.9970	**.0057**	.0145	.0233	.0323	.0413	15	29	44	58	73
64	2.0503	.0594	.0686	.0778	.0872	.0965	.1060	.1155	.1251	.1348	16	31	47	63	78
65	2.1445	.1543	.1642	.1742	.1842	.1943	.2045	.2148	.2251	.2355	17	34	51	68	85
66	2.2460	.2566	.2673	.2781	.2889	.2998	.3109	.3220	.3332	.3445	18	37	55	73	92
67	2.3559	.3673	.3789	.3906	.4023	.4142	.4262	.4383	.4504	.4627	20	40	60	79	99
68	2.4751	.4876	.5002	.5129	.5257	.5386	.5517	.5649	.5782	.5916	22	43	65	87	108
69	2.6051	.6187	.6325	.6464	.6605	.6746	.6889	.7034	.7179	.7326	24	47	71	95	119
70	2.7475	.7625	.7776	.7929	.8083	.8239	.8397	.8556	.8716	.8878	26	52	78	104	130
71	2.9042	.9208	.9375	.9544	.9714	.9887	**.0061**	.0237	.0415	.0595	29	58	87	116	144
72	3.0777	.0961	.1146	.1334	.1524	.1716	.1910	.2106	.2305	.2506	32	64	96	129	161
73	3.2709	.2914	.3122	.3332	.3544	.3759	.3977	.4197	.4420	.4646	36	72	108	144	180
74	3.4874	.5105	.5339	.5576	.5816	.6059	.6305	.6554	.6806	.7062	41	81	122	163	204
75	3.7321	.7583	.7848	.8118	.8391	.8667	.8947	.9232	.9520	.9812					
76	4.0108	.0408	.0713	.1022	.1335	.1653	.1976	.2303	.2635	.2972					
77	4.3315	.3662	.4015	.4373	.4737	.5107	.5483	.5864	.6252	.6646					
78	4.7046	.7453	.7867	.8288	.8716	.9152	.9594	**.0045**	.0504	.0970	Use ordinary interpolation.				
79	5.1446	.1929	.2422	.2924	.3435	.3955	.4486	.5026	.5578	.6140					
80	5.6713	.7297	.7894	.8502	.9124	.9758	**.0405**	.1066	.1742	.2432					
81	6.3138	.3859	.4596	.5350	.6122	.6912	.7720	.8548	.9395	**.0264**					
82	7.1154	.2066	.3002	.3962	.4947	.5958	.6996	.8062	.9158	**.0285**					
83	8.1443	.2636	.3863	.5126	.6427	.7769	.9152	**.0579**	.2052	.3572					
84	9.5144	.6768	.8448	**.0187**	.1988	.3854	.5789	.7797	.9882	**.2048**					
85	11.430	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95					
86	14.301	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46					
87	19.081	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27					
88	28.636	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08					
89	57.290	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0					

Heavy-face type indicates that the integral part is to be increased by 1.

LOGARITHMS OF SINES. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	−∞	$\bar{3}$.242	.5429	.7190	.8439	.9408	**.0200**	.0869	.1449	.1961					
1	$\bar{2}$.2419	.2832	.3210	.3558	.3880	.4179	.4459	.4723	.4971	.5206	Use ordinary interpolation.				
2	$\bar{2}$.5428	.5640	.5842	.6035	.6220	.6397	.6567	.6731	.6889	.7041					
3	$\bar{2}$.7188	.7330	.7468	.7602	.7731	.7857	.7979	.8098	.8213	.8326					
4	$\bar{2}$.8436	.8543	.8647	.8749	.8849	.8946	.9042	.9135	.9226	.9315					
5	$\bar{2}$.9403	.9489	.9573	.9655	.9736	.9816	.9894	.9970	**.0046**	.0120	13	26	39	52	65
6	$\bar{1}$.0192	.0264	.0334	.0403	.0472	.0539	.0605	.0670	.0734	.0797	11	22	33	44	55
7	$\bar{1}$.0859	.0920	.0981	.1040	.1099	.1157	.1214	.1271	.1326	.1381	10	19	29	38	48
8	$\bar{1}$.1436	.1489	.1542	.1594	.1646	.1697	.1747	.1797	.1847	.1895	8	17	25	34	42
9	$\bar{1}$.1943	.1991	.2038	.2085	.2131	.2176	.2221	.2266	.2310	.2353	8	15	23	30	38
10	$\bar{1}$.2397	.2439	.2482	.2524	.2565	.2606	.2647	.2687	.2727	.2767	7	14	20	27	34
11	$\bar{1}$.2806	.2845	.2883	.2921	.2959	.2997	.3034	.3070	.3107	.3143	6	12	19	25	31
12	$\bar{1}$.3179	.3214	.3250	.3284	.3319	.3353	.3387	.3421	.3455	.3488	6	11	17	23	28
13	$\bar{1}$.3521	.3554	.3586	.3618	.3650	.3682	.3713	.3745	.3775	.3806	5	11	16	21	26
14	$\bar{1}$.3837	.3867	.3897	.3927	.3957	.3986	.4015	.4044	.4073	.4102	5	10	15	20	24
15	$\bar{1}$.4130	.4158	.4186	.4214	.4242	.4269	.4296	.4323	.4350	.4377	5	9	14	18	23
16	$\bar{1}$.4403	.4430	.4456	.4482	.4508	.4533	.4559	.4584	.4609	.4634	4	9	13	17	21
17	$\bar{1}$.4659	.4684	.4709	.4733	.4757	.4781	.4805	.4829	.4853	.4876	4	8	12	16	20
18	$\bar{1}$.4900	.4923	.4946	.4969	.4992	.5015	.5037	.5060	.5082	.5104	4	8	11	15	19
19	$\bar{1}$.5126	.5148	.5170	.5192	.5213	.5235	.5256	.5278	.5299	.5320	4	7	11	14	18
20	$\bar{1}$.5341	.5361	.5382	.5402	.5423	.5443	.5463	.5484	.5504	.5523	3	7	10	14	17
21	$\bar{1}$.5543	.5563	.5583	.5602	.5621	.5641	.5660	.5679	.5698	.5717	3	6	10	13	16
22	$\bar{1}$.5736	.5754	.5773	.5792	.5810	.5828	.5847	.5865	.5883	.5901	3	6	9	12	15
23	$\bar{1}$.5919	.5937	.5954	.5972	.5990	.6007	.6024	.6042	.6059	.6076	3	6	9	12	15
24	$\bar{1}$.6093	.6110	.6127	.6144	.6161	.6177	.6194	.6210	.6227	.6243	3	6	8	11	14
25	$\bar{1}$.6259	.6276	.6292	.6308	.6324	.6340	.6356	.6371	.6387	.6403	3	5	8	11	13
26	$\bar{1}$.6418	.6434	.6449	.6465	.6480	.6495	.6510	.6526	.6541	.6556	3	5	8	10	13
27	$\bar{1}$.6570	.6585	.6600	.6615	.6629	.6644	.6659	.6673	.6687	.6702	2	5	7	10	12
28	$\bar{1}$.6716	.6730	.6744	.6759	.6773	.6787	.6801	.6814	.6828	.6842	2	5	7	9	12
29	$\bar{1}$.6856	.6869	.6883	.6896	.6910	.6923	.6937	.6950	.6963	.6977	2	4	7	9	11
30	$\bar{1}$.6990	.7003	.7016	.7029	.7042	.7055	.7068	.7080	.7093	.7106	2	4	6	9	11
31	$\bar{1}$.7118	.7131	.7144	.7156	.7168	.7181	.7193	.7205	.7218	.7230	2	4	6	8	10
32	$\bar{1}$.7242	.7254	.7266	.7278	.7290	.7302	.7314	.7326	.7338	.7349	2	4	6	8	10
33	$\bar{1}$.7361	.7373	.7384	.7396	.7407	.7419	.7430	.7442	.7453	.7464	2	4	6	8	10
34	$\bar{1}$.7476	.7487	.7498	.7509	.7520	.7531	.7542	.7553	.7564	.7575	2	4	6	7	9
35	$\bar{1}$.7586	.7597	.7607	.7618	.7629	.7640	.7650	.7661	.7671	.7682	2	4	5	7	9
36	$\bar{1}$.7692	.7703	.7713	.7723	.7734	.7744	.7754	.7764	.7774	.7785	2	3	5	7	9
37	$\bar{1}$.7795	.7805	.7815	.7825	.7835	.7844	.7854	.7864	.7874	.7884	2	3	5	7	8
38	$\bar{1}$.7893	.7903	.7913	.7922	.7932	.7941	.7951	.7960	.7970	.7979	2	3	5	6	8
39	$\bar{1}$.7989	.7998	.8007	.8017	.8026	.8035	.8044	.8053	.8063	.8072	2	3	5	6	8
40	$\bar{1}$.8081	.8090	.8099	.8108	.8117	.8125	.8134	.8143	.8152	.8161	1	3	4	6	7
41	$\bar{1}$.8169	.8178	.8187	.8195	.8204	.8213	.8221	.8230	.8238	.8247	1	3	4	6	7
42	$\bar{1}$.8255	.8264	.8272	.8280	.8289	.8297	.8305	.8313	.8322	.8330	1	3	4	6	7
43	$\bar{1}$.8338	.8346	.8354	.8362	.8370	.8378	.8386	.8394	.8402	.8410	1	3	4	5	7
44	$\bar{1}$.8418	.8426	.8433	.8441	.8449	.8457	.8464	.8472	.8480	.8487	1	3	4	5	6

Heavy-face type indicates that the characteristic is *numerically* 1 less.

LOGARITHMS OF SINES. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	Ī.8495	.8502	.8510	.8517	.8525	.8532	.8540	.8547	.8555	.8562	1	2	4	5	6
46	Ī.8569	.8577	.8584	.8591	.8598	.8606	.8613	.8620	.8627	.8634	1	2	4	5	6
47	Ī.8641	.8648	.8655	.8662	.8669	.8676	.8683	.8690	.8697	.8704	1	2	3	5	6
48	Ī.8711	.8718	.8724	.8731	.8738	.8745	.8751	.8758	.8765	.8771	1	2	3	4	6
49	Ī.8778	.8784	.8791	.8797	.8804	.8810	.8817	.8823	.8830	.8836	1	2	3	4	5
50	Ī.8843	.8849	.8855	.8862	.8868	.8874	.8880	.8887	.8893	.8899	1	2	3	4	5
51	Ī.8905	.8911	.8917	.8923	.8929	.8935	.8941	.8947	.8953	.8959	1	2	3	4	5
52	Ī.8965	.8971	.8977	.8983	.8989	.8995	.9000	.9006	.9012	.9018	1	2	3	4	5
53	Ī.9023	.9029	.9035	.9041	.9046	.9052	.9057	.9063	.9069	.9074	1	2	3	4	5
54	Ī.9080	.9085	.9091	.9096	.9101	.9107	.9112	.9118	.9123	.9128	1	2	3	4	5
55	Ī.9134	.9139	.9144	.9149	.9155	.9160	.9165	.9170	.9175	.9181	1	2	3	3	4
56	Ī.9186	.9191	.9196	.9201	.9206	.9211	.9216	.9221	.9226	.9231	1	2	3	3	4
57	Ī.9236	.9241	.9246	.9251	.9255	.9260	.9265	.9270	.9275	.9279	1	2	2	3	4
58	Ī.9284	.9289	.9294	.9298	.9303	.9308	.9312	.9317	.9322	.9326	1	2	2	3	4
59	Ī.9331	.9335	.9340	.9344	.9349	.9353	.9358	.9362	.9367	.9371	1	1	2	3	4
60	Ī.9375	.9380	.9384	.9388	.9393	.9397	.9401	.9406	.9410	.9414	1	1	2	3	4
61	Ī.9418	.9422	.9427	.9431	.9435	.9439	.9443	.9447	.9451	.9455	1	1	2	3	3
62	Ī.9459	.9463	.9467	.9471	.9475	.9479	.9483	.9487	.9491	.9495	1	1	2	3	3
63	Ī.9499	.9503	.9507	.9510	.9514	.9518	.9522	.9525	.9529	.9533	1	1	2	3	3
64	Ī.9537	.9540	.9544	.9548	.9551	.9555	.9558	.9562	.9566	.9569	1	1	2	2	3
65	Ī.9573	.9576	.9580	.9583	.9587	.9590	.9594	.9597	.9601	.9604	1	1	2	2	3
66	Ī.9607	.9611	.9614	.9617	.9621	.9624	.9627	.9631	.9634	.9637	1	1	2	2	3
67	Ī.9640	.9643	.9647	.9650	.9653	.9656	.9659	.9662	.9666	.9669	1	1	2	2	3
68	Ī.9672	.9675	.9678	.9681	.9684	.9687	.9690	.9693	.9696	.9699	0	1	1	2	2
69	Ī.9702	.9704	.9707	.9710	.9713	.9716	.9719	.9722	.9724	.9727	0	1	1	2	2
70	Ī.9730	.9733	.9735	.9738	.9741	.9743	.9746	.9749	.9751	.9754	0	1	1	2	2
71	Ī.9757	.9759	.9762	.9764	.9767	.9770	.9772	.9775	.9777	.9780	0	1	1	2	2
72	Ī.9782	.9785	.9787	.9789	.9792	.9794	.9797	.9799	.9801	.9804	0	1	1	2	2
73	Ī.9806	.9808	.9811	.9813	.9815	.9817	.9820	.9822	.9824	.9826	0	1	1	2	2
74	Ī.9828	.9831	.9833	.9835	.9837	.9839	.9841	.9843	.9845	.9847	0	1	1	1	2
75	Ī.9849	.9851	.9853	.9855	.9857	.9859	.9861	.9863	.9865	.9867	0	1	1	1	2
76	Ī.9869	.9871	.9873	.9875	.9876	.9878	.9880	.9882	.9884	.9885	0	1	1	1	2
77	Ī.9887	.9889	.9891	.9892	.9894	.9896	.9897	.9899	.9901	.9902	0	1	1	1	1
78	Ī.9904	.9906	.9907	.9909	.9910	.9912	.9913	.9915	.9916	.9918	0	1	1	1	1
79	Ī.9919	.9921	.9922	.9924	.9925	.9927	.9928	.9929	.9931	.9932	0	0	1	1	1
80	Ī.9934	.9935	.9936	.9937	.9939	.9940	.9941	.9943	.9944	.9945	0	0	1	1	1
81	Ī.9946	.9947	.9949	.9950	.9951	.9952	.9953	.9954	.9955	.9956	0	0	1	1	1
82	Ī.9958	.9959	.9960	.9961	.9962	.9963	.9964	.9965	.9966	.9967	0	0	1	1	1
83	Ī.9968	.9968	.9969	.9970	.9971	.9972	.9973	.9974	.9975	.9975	0	0	0	1	1
84	Ī.9976	.9977	.9978	.9978	.9979	.9980	.9981	.9981	.9982	.9983	0	0	0	0	1
85	Ī.9983	.9984	.9985	.9985	.9986	.9987	.9987	.9988	.9988	.9989	0	0	0	0	0
86	Ī.9989	.9990	.9990	.9991	.9991	.9992	.9992	.9993	.9993	.9994	0	0	0	0	0
87	Ī.9994	.9994	.9995	.9995	.9996	.9996	.9996	.9996	.9997	.9997	0	0	0	0	0
88	Ī.9997	.9998	.9998	.9998	.9998	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0
89	Ī.9999	.9999	**.0000**	.0000	.0000	.0000	.0000	.0000	.0000	.0000	0	0	0	0	0

Heavy-face type indicates that the characteristic is *numerically* 1 less.

LOGARITHMS OF COSINES. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	— Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	0.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	**.9999**	0	0	0	0	0
1	$\bar{1}$.9999	.9999	.9999	.9999	.9999	.9999	.9998	.9998	.9998	.9998	0	0	0	0	0
2	$\bar{1}$.9997	.9997	.9997	.9997	.9996	.9996	.9996	.9995	.9995	.9994	0	0	0	0	0
3	$\bar{1}$.9994	.9994	.9993	.9993	.9992	.9992	.9991	.9991	.9990	.9990	0	0	0	0	0
4	$\bar{1}$.9989	.9989	.9988	.9988	.9987	.9987	.9986	.9985	.9985	.9984	0	0	0	0	0
5	$\bar{1}$.9983	.9983	.9982	.9981	.9981	.9980	.9979	.9978	.9978	.9977	0	0	0	0	1
6	$\bar{1}$.9976	.9975	.9975	.9974	.9973	.9972	.9971	.9970	.9969	.9968	0	0	0	1	1
7	$\bar{1}$.9968	.9967	.9966	.9965	.9964	.9963	.9962	.9961	.9960	.9959	0	0	1	1	1
8	$\bar{1}$.9958	.9956	.9955	.9954	.9953	.9952	.9951	.9950	.9949	.9947	0	0	1	1	1
9	$\bar{1}$.9946	.9945	.9944	.9943	.9941	.9940	.9939	.9937	.9936	.9935	0	0	1	1	1
10	$\bar{1}$.9934	.9932	.9931	.9929	.9928	.9927	.9925	.9924	.9922	.9921	0	0	1	1	1
11	$\bar{1}$.9919	.9918	.9916	.9915	.9913	.9912	.9910	.9909	.9907	.9906	0	1	1	1	1
12	$\bar{1}$.9904	.9902	.9901	.9899	.9897	.9896	.9894	.9892	.9891	.9889	0	1	1	1	1
13	$\bar{1}$.9887	.9885	.9884	.9882	.9880	.9878	.9876	.9875	.9873	.9871	0	1	1	1	2
14	$\bar{1}$.9869	.9867	.9865	.9863	.9861	.9859	.9857	.9855	.9853	.9851	0	1	1	1	2
15	$\bar{1}$.9849	.9847	.9845	.9843	.9841	.9839	.9837	.9835	.9833	.9831	0	1	1	1	2
16	$\bar{1}$.9828	.9826	.9824	.9822	.9820	.9817	.9815	.9813	.9811	.9808	0	1	1	2	2
17	$\bar{1}$.9806	.9804	.9801	.9799	.9797	.9794	.9792	.9789	.9787	.9785	0	1	1	2	2
18	$\bar{1}$.9782	.9780	.9777	.9775	.9772	.9770	.9767	.9764	.9762	.9759	0	1	1	2	2
19	$\bar{1}$.9757	.9754	.9751	.9749	.9746	.9743	.9741	.9738	.9735	.9733	0	1	1	2	2
20	$\bar{1}$.9730	.9727	.9724	.9722	.9719	.9716	.9713	.9710	.9707	.9704	0	1	1	2	2
21	$\bar{1}$.9702	.9699	.9696	.9693	.9690	.9687	.9684	.9681	.9678	.9675	0	1	1	2	2
22	$\bar{1}$.9672	.9669	.9666	.9662	.9659	.9656	.9653	.9650	.9647	.9643	1	1	2	2	3
23	$\bar{1}$.9640	.9637	.9634	.9631	.9627	.9624	.9621	.9617	.9614	.9611	1	1	2	2	3
24	$\bar{1}$.9607	.9604	.9601	.9597	.9594	.9590	.9587	.9583	.9580	.9576	1	1	2	2	3
25	$\bar{1}$.9573	.9569	.9566	.9562	.9558	.9555	.9551	.9548	.9544	.9540	1	1	2	2	3
26	$\bar{1}$.9537	.9533	.9529	.9525	.9522	.9518	.9514	.9510	.9507	.9503	1	1	2	3	3
27	$\bar{1}$.9499	.9495	.9491	.9487	.9483	.9479	.9475	.9471	.9467	.9463	1	1	2	3	3
28	$\bar{1}$.9459	.9455	.9451	.9447	.9443	.9439	.9435	.9431	.9427	.9422	1	1	2	3	3
29	$\bar{1}$.9418	.9414	.9410	.9406	.9401	.9397	.9393	.9388	.9384	.9380	1	1	2	3	4
30	$\bar{1}$.9375	.9371	.9367	.9362	.9358	.9353	.9349	.9344	.9340	.9335	1	1	2	3	4
31	$\bar{1}$.9331	.9326	.9322	.9317	.9312	.9308	.9303	.9298	.9294	.9289	1	2	2	3	4
32	$\bar{1}$.9284	.9279	.9275	.9270	.9265	.9260	.9255	.9251	.9246	.9241	1	2	2	3	4
33	$\bar{1}$.9236	.9231	.9226	.9221	.9216	.9211	.9206	.9201	.9196	.9191	1	2	3	3	4
34	$\bar{1}$.9186	.9181	.9175	.9170	.9165	.9160	.9155	.9149	.9144	.9139	1	2	3	3	4
35	$\bar{1}$.9134	.9128	.9123	.9118	.9112	.9107	.9101	.9096	.9091	.9085	1	2	3	4	5
36	$\bar{1}$.9080	.9074	.9069	.9063	.9057	.9052	.9046	.9041	.9035	.9029	1	2	3	4	5
37	$\bar{1}$.9023	.9018	.9012	.9006	.9000	.8995	.8989	.8983	.8977	.8971	1	2	3	4	5
38	$\bar{1}$.8965	.8959	.8953	.8947	.8941	.8935	.8929	.8923	.8917	.8911	1	2	3	4	5
39	$\bar{1}$.8905	.8899	.8893	.8887	.8880	.8874	.8868	.8862	.8855	.8849	1	2	3	4	5
40	$\bar{1}$.8843	.8836	.8830	.8823	.8817	.8810	.8804	.8797	.8791	.8784	**1**	2	3	4	5
41	$\bar{1}$.8778	.8771	.8765	.8758	.8751	.8745	.8738	.8731	.8724	.8718	**1**	2	3	5	6
42	$\bar{1}$.8711	.8704	.8697	.8690	.8683	.8676	.8669	.8662	.8655	.8648	**1**	2	3	5	6
43	$\bar{1}$.8641	.8634	.8627	.8620	.8613	.8606	.8598	.8591	.8584	.8577	1	2	4	5	6
44	$\bar{1}$.8569	.8562	.8555	.8547	.8540	.8532	.8525	.8517	.8510	.8502	1	2	4	5	6

Heavy-face type indicates that the characteristic is *numerically* 1 greater.

LOGARITHMS OF COSINES. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	— Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	Ī.8495	.8487	.8480	.8472	.8464	.8457	.8449	.8441	.8433	.8426	1	3	4	5	6
46	Ī.8418	.8410	.8402	.8394	.8386	.8378	.8370	.8362	.8354	.8346	1	3	4	5	7
47	Ī.8338	.8330	.8322	.8313	.8305	.8297	.8289	.8280	.8272	.8264	1	3	4	6	7
48	Ī.8255	.8247	.8238	.8230	.8221	.8213	.8204	.8195	.8187	.8178	1	3	4	6	7
49	Ī.8169	.8161	.8152	.8143	.8134	.8125	.8117	.8108	.8099	.8090	1	3	4	6	7
50	Ī.8081	.8072	.8063	.8053	.8044	.8035	.8026	.8017	.8007	.7998	2	3	5	6	8
51	Ī.7989	.7979	.7970	.7960	.7951	.7941	.7932	.7922	.7913	.7903	2	3	5	6	8
52	Ī.7893	.7884	.7874	.7864	.7854	.7844	.7835	.7825	.7815	.7805	2	3	5	7	8
53	Ī.7795	.7785	.7774	.7764	.7754	.7744	.7734	.7723	.7713	.7703	2	3	5	7	9
54	Ī.7692	.7682	.7671	.7661	.7650	.7640	.7629	.7618	.7607	.7597	2	4	5	7	9
55	Ī.7586	.7575	.7564	.7553	.7542	.7531	.7520	.7509	.7498	.7487	2	4	6	7	9
56	Ī.7476	.7464	.7453	.7442	.7430	.7419	.7407	.7396	.7384	.7373	2	4	6	8	10
57	Ī.7361	.7349	.7338	.7326	.7314	.7302	.7290	.7278	.7266	.7254	2	4	6	8	10
58	Ī.7242	.7230	.7218	.7205	.7193	.7181	.7168	.7156	.7144	.7131	2	4	6	8	10
59	Ī.7118	.7106	.7093	.7080	.7068	.7055	.7042	.7029	.7016	.7003	2	4	6	9	11
60	Ī.6990	.6977	.6963	.6950	.6937	.6923	.6910	.6896	.6883	.6869	2	4	7	9	11
61	Ī.6856	.6842	.6828	.6814	.6801	.6787	.6773	.6759	.6744	.6730	2	5	7	9	12
62	Ī.6716	.6702	.6687	.6673	.6659	.6644	.6629	.6615	.6600	.6585	2	5	7	10	12
63	Ī.6570	.6556	.6541	.6526	.6510	.6495	.6480	.6465	.6449	.6434	3	5	8	10	13
64	Ī.6418	.6403	.6387	.6371	.6356	.6340	.6324	.6308	.6292	.6276	3	5	8	11	13
65	Ī.6259	.6243	.6227	.6210	.6194	.6177	.6161	.6144	.6127	.6110	3	6	8	11	14
66	Ī.6093	.6076	.6059	.6042	.6024	.6007	.5990	.5972	.5954	.5937	3	6	9	12	15
67	Ī.5919	.5901	.5883	.5865	.5847	.5828	.5810	.5792	.5773	.5754	3	6	9	12	15
68	Ī.5736	.5717	.5698	.5679	.5660	.5641	.5621	.5602	.5583	.5563	3	6	10	13	16
69	Ī.5543	.5523	.5504	.5484	.5463	.5443	.5423	.5402	.5382	.5361	3	7	10	14	17
70	Ī.5341	.5320	.5299	.5278	.5256	.5235	.5213	.5192	.5170	.5148	4	7	11	14	18
71	Ī.5126	.5104	.5082	.5060	.5037	.5015	.4992	.4969	.4946	.4923	4	8	11	15	19
72	Ī.4900	.4876	.4853	.4829	.4805	.4781	.4757	.4733	.4709	.4684	4	8	12	16	20
73	Ī.4659	.4634	.4609	.4584	.4559	.4533	.4508	.4482	.4456	.4430	4	9	13	17	21
74	Ī.4403	.4377	.4350	.4323	.4296	.4269	.4242	.4214	.4186	.4158	5	9	14	18	23
75	Ī.4130	.4102	.4073	.4044	.4015	.3986	.3957	.3927	.3897	.3867	5	10	15	20	24
76	Ī.3837	.3806	.3775	.3745	.3713	.3682	.3650	.3618	.3586	.3554	5	11	16	21	26
77	Ī.3521	.3488	.3455	.3421	.3387	.3353	.3319	.3284	.3250	.3214	6	11	17	23	28
78	Ī.3179	.3143	.3107	.3070	.3034	.2997	.2959	.2921	.2883	.2845	6	12	19	25	31
79	Ī.2806	.2767	.2727	.2687	.2647	.2606	.2565	.2524	.2482	.2439	7	14	20	27	34
80	Ī.2397	.2353	.2310	.2266	.2221	.2176	.2131	.2085	.2038	.1991	8	15	23	30	38
81	Ī.1943	.1895	.1847	.1797	.1747	.1697	.1646	.1594	.1542	.1489	8	17	25	34	42
82	Ī.1436	.1381	.1326	.1271	.1214	.1157	.1099	.1040	.0981	.0920	10	19	29	38	48
83	Ī.0859	.0797	.0734	.0670	.0605	.0539	.0472	.0403	.0334	.0264	11	22	33	44	55
84	Ī.0192	.0120	.0046	**.9970**	.9894	.9816	.9736	.9655	.9573	.9489	13	26	39	52	66
85	$\bar{2}$.9403	.9315	.9226	.9135	.9042	.8946	.8849	.8749	.8647	.8543	16	32	48	64	80
86	$\bar{2}$.8436	.8326	.8213	.8098	.7979	.7857	.7731	.7602	.7468	.7330	Use ordinary interpolation.				
87	$\bar{2}$.7188	.7041	.6889	.6731	.6567	.6397	.6220	.6035	.5842	.5640					
88	$\bar{2}$.5428	.5206	.4971	.4723	.4459	.4179	.3880	.3558	.3210	.2832					
89	$\bar{2}$.2419	.1961	.1449	.0869	.0200	**.9408**	.8439	.7190	.5429	.2419					

Heavy-face type indicates that the characteristic is *numerically* 1 greater.

LOGARITHMS OF TANGENTS. 0°–45°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	−∞	3.242	.5429	.7190	.8439	.9408	**.0200**	.0870	.1450	.1962					
1	2.2419	.2833	.3211	.3559	.3881	.4181	.4461	.4725	.4973	.5208	Use ordinary interpolation.				
2	2.5431	.5643	.5845	.6038	.6223	.6401	.6571	.6736	.6894	.7046					
3	2.7194	.7337	.7475	.7609	.7739	.7865	.7988	.8107	.8223	.8336					
4	2.8446	.8554	.8659	.8762	.8862	.8960	.9056	.9150	.9241	.9331	16	32	48	64	81
5	2.9420	.9506	.9591	.9674	.9756	.9836	.9915	.9992	**.0068**	.0143	13	26	40	53	66
6	1.0216	.0289	.0360	.0430	.0499	.0567	.0633	.0699	.0764	.0828	11	22	34	45	56
7	1.0891	.0954	.1015	.1076	.1135	.1194	.1252	.1310	.1367	.1423	10	20	29	39	49
8	1.1478	.1533	.1587	.1640	.1693	.1745	.1797	.1848	.1898	.1948	9	17	26	35	43
9	1.1997	.2046	.2094	.2142	.2189	.2236	.2282	.2328	.2374	.2419	8	16	23	31	39
10	1.2463	.2507	.2551	.2594	.2637	.2680	.2722	.2764	.2805	.2846	7	14	21	28	35
11	1.2887	.2927	.2967	.3006	.3046	.3085	.3123	.3162	.3200	.3237	6	13	19	26	32
12	1.3275	.3312	.3349	.3385	.3422	.3458	.3493	.3529	.3564	.3599	6	12	18	24	30
13	1.3634	.3668	.3702	.3736	.3770	.3804	.3837	.3870	.3903	.3935	6	11	17	22	28
14	1.3968	.4000	.4032	.4064	.4095	.4127	.4158	.4189	.4220	.4250	5	10	16	21	26
15	1.4281	.4311	.4341	.4371	.4400	.4430	.4459	.4488	.4517	.4546	5	10	15	20	25
16	1.4575	.4603	.4632	.4660	.4688	.4716	.4744	.4771	.4799	.4826	5	9	14	19	23
17	1.4853	.4880	.4907	.4934	.4961	.4987	.5014	.5040	.5066	.5092	4	9	13	18	22
18	1.5118	.5143	.5169	.5195	.5220	.5245	.5270	.5295	.5320	.5345	4	8	13	17	21
19	1.5370	.5394	.5419	.5443	.5467	.5491	.5516	.5539	.5563	.5587	4	8	12	16	20
20	1.5611	.5634	.5658	.5681	.5704	.5727	.5750	.5773	.5796	.5819	4	8	12	15	19
21	1.5842	.5864	.5887	.5909	.5932	.5954	.5976	.5998	.6020	.6042	4	7	11	15	19
22	1.6064	.6086	.6108	.6129	.6151	.6172	.6194	.6215	.6236	.6257	4	7	11	14	18
23	1.6279	.6300	.6321	.6341	.6362	.6383	.6404	.6424	.6445	.6465	3	7	10	14	17
24	1.6486	.6506	.6527	.6547	.6567	.6587	.6607	.6627	.6647	.6667	3	7	10	13	17
25	1.6687	.6706	.6726	.6746	.6765	.6785	.6804	.6824	.6843	.6863	3	7	10	13	16
26	1.6882	.6901	.6920	.6939	.6958	.6977	.6996	.7015	.7034	.7053	3	6	9	13	16
27	1.7072	.7090	.7109	.7128	.7146	.7165	.7183	.7202	.7220	.7238	3	6	9	12	15
28	1.7257	.7275	.7293	.7311	.7330	.7348	.7366	.7384	.7402	.7420	3	6	9	12	15
29	1.7438	.7455	.7473	.7491	.7509	.7526	.7544	.7562	.7579	.7597	3	6	9	12	15
30	1.7614	.7632	.7649	.7667	.7684	.7701	.7719	.7736	.7753	.7771	3	6	9	12	14
31	1.7788	.7805	.7822	.7839	.7856	.7873	.7890	.7907	.7924	.7941	3	6	9	11	14
32	1.7958	.7975	.7992	.8008	.8025	.8042	.8059	.8075	.8092	.8109	3	6	8	11	14
33	1.8125	.8142	.8158	.8175	.8191	.8208	.8224	.8241	.8257	.8274	3	5	8	11	14
34	1.8290	.8306	.8323	.8339	.8355	.8371	.8388	.8404	.8420	.8436	3	5	8	11	14
35	1.8452	.8468	.8484	.8501	.8517	.8533	.8549	.8565	.8581	.8597	3	5	8	11	13
36	1.8613	.8629	.8644	.8660	.8676	.8692	.8708	.8724	.8740	.8755	3	5	8	11	13
37	1.8771	.8787	.8803	.8818	.8834	.8850	.8865	.8881	.8897	.8912	3	5	8	10	13
38	1.8928	.8944	.8959	.8975	.8990	.9006	.9022	.9037	.9053	.9068	3	5	8	10	13
39	1.9084	.9099	.9115	.9130	.9146	.9161	.9176	.9192	.9207	.9223	3	5	8	10	13
40	1.9238	.9254	.9269	.9284	.9300	.9315	.9330	.9346	.9361	.9376	3	5	8	10	13
41	1.9392	.9407	.9422	.9438	.9453	.9468	.9483	.9499	.9514	.9529	3	5	8	10	13
42	1.9544	.9560	.9575	.9590	.9605	.9621	.9636	.9651	.9666	.9681	3	5	8	10	13
43	1.9697	.9712	.9727	.9742	.9757	.9773	.9788	.9803	.9818	.9833	3	5	8	10	13
44	1.9848	.9864	.9879	.9894	.9909	.9924	.9939	.9955	.9970	.9985	3	5	8	10	13

Heavy-face type indicates that the characteristic is *numerically* 1 less.

LOGARITHMS OF TANGENTS. 45°–90°

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	0.0000	.0015	.0030	.0045	.0061	.0076	.0091	.0106	.0121	.0136	3	5	8	10	13
46	0.0152	.0167	.0182	.0197	.0212	.0228	.0243	.0258	.0273	.0288	3	5	8	10	13
47	0.0303	.0319	.0334	.0349	.0364	.0379	.0395	.0410	.0425	.0440	3	5	8	10	13
48	0.0456	.0471	.0486	.0501	.0517	.0532	.0547	.0562	.0578	.0593	3	5	8	10	13
49	0.0608	.0624	.0639	.0654	.0670	.0685	.0700	.0716	.0731	.0746	3	5	8	10	13
50	0.0762	.0777	.0793	.0808	.0824	.0839	.0854	.0870	.0885	.0901	3	5	8	10	13
51	0.0916	.0932	.0947	.0963	.0978	.0994	.1010	.1025	.1041	.1056	3	5	8	10	13
52	0.1072	.1088	.1103	.1119	.1135	.1150	.1166	.1182	.1197	.1213	3	5	8	10	13
53	0.1229	.1245	.1260	.1276	.1292	.1308	.1324	.1340	.1356	.1371	3	5	8	11	13
54	0.1387	.1403	.1419	.1435	.1451	.1467	.1483	.1499	.1516	.1532	3	5	8	11	13
55	0.1548	.1564	.1580	.1596	.1612	.1629	.1645	.1661	.1677	.1694	3	5	8	11	14
56	0.1710	.1726	.1743	.1759	.1776	.1792	.1809	.1825	.1842	.1858	3	5	8	11	14
57	0.1875	.1891	.1908	.1925	.1941	.1958	.1975	.1992	.2008	.2025	3	6	8	11	14
58	0.2042	.2059	.2076	.2093	.2110	.2127	.2144	.2161	.2178	.2195	3	6	9	11	14
59	0.2212	.2229	.2247	.2264	.2281	.2299	.2316	.2333	.2351	.2368	3	6	9	12	14
60	0.2386	.2403	.2421	.2438	.2456	.2474	.2491	.2509	.2527	.2545	3	6	9	12	15
61	0.2562	.2580	.2598	.2616	.2634	.2652	.2670	.2689	.2707	.2725	3	6	9	12	15
62	0.2743	.2762	.2780	.2798	.2817	.2835	.2854	.2872	.2891	.2910	3	6	9	12	15
63	0.2928	.2947	.2966	.2985	.3004	.3023	.3042	.3061	.3080	.3099	3	6	9	13	16
64	0.3118	.3137	.3157	.3176	.3196	.3215	.3235	.3254	.3274	.3294	3	7	10	13	16
65	0.3313	.3333	.3353	.3373	.3393	.3413	.3433	.3453	.3473	.3494	3	7	10	13	17
66	0.3514	.3535	.3555	.3576	.3596	.3617	.3638	.3659	.3679	.3700	3	7	10	14	17
67	0.3721	.3743	.3764	.3785	.3806	.3828	.3849	.3871	.3892	.3914	4	7	11	14	18
68	0.3936	.3958	.3980	.4002	.4024	.4046	.4068	.4091	.4113	.4136	4	7	11	15	19
69	0.4158	.4181	.4204	.4227	.4250	.4273	.4296	.4319	.4342	.4366	4	8	12	15	19
70	0.4389	.4413	.4437	.4461	.4484	.4509	.4533	.4557	.4581	.4606	4	8	12	16	20
71	0.4630	.4655	.4680	.4705	.4730	.4755	.4780	.4805	.4831	.4857	4	8	13	17	21
72	0.4882	.4908	.4934	.4960	.4986	.5013	.5039	.5066	.5093	.5120	4	9	13	18	22
73	0.5147	.5174	.5201	.5229	.5256	.5284	.5312	.5340	.5368	.5397	5	9	14	19	23
74	0.5425	.5454	.5483	.5512	.5541	.5570	.5600	.5629	.5659	.5689	5	10	15	20	25
75	0.5719	.5750	.5780	.5811	.5842	.5873	.5905	.5936	.5968	.6000	5	10	16	21	26
76	0.6032	.6065	.6097	.6130	.6163	.6196	.6230	.6264	.6298	.6332	6	11	17	22	28
77	0.6366	.6401	.6436	.6471	.6507	.6542	.6578	.6615	.6651	.6688	6	12	18	24	30
78	0.6725	.6763	.6800	.6838	.6877	.6915	.6954	.6994	.7033	.7073	6	13	19	26	32
79	0.7113	.7154	.7195	.7236	.7278	.7320	.7363	.7406	.7449	.7493	7	14	21	28	35
80	0.7537	.7581	.7626	.7672	.7718	.7764	.7811	.7858	.7906	.7954	8	16	21	31	39
81	0.8003	.8052	.8102	.8152	.8203	.8255	.8307	.8360	.8413	.8467	9	17	26	35	43
82	0.8522	.8577	.8633	.8690	.8748	.8806	.8865	.8924	.8985	.9046	10	20	29	39	49
83	0.9109	.9172	.9236	.9301	.9367	.9433	.9501	.9570	.9640	.9711	11	22	34	45	56
84	0.9784	.9857	.9932	**.0008**	.0085	.0164	.0244	.0326	.0409	.0494	13	26	40	53	66
85	1.0580	.0669	.0759	.0850	.0944	.1040	.1138	.1238	.1341	.1446	16	32	48	65	81
86	1.1554	.1664	.1777	.1893	.2012	.2135	.2261	.2391	.2525	.2663	Use ordinary interpolation.				
87	1.2806	.2954	.3106	.3264	.3429	.3599	.3777	.3962	.4155	.4357					
88	1.4569	.4792	.5027	.5275	.5539	.5819	.6119	.6441	.6789	.7167					
89	1.7581	.8038	.8550	.9130	.9800	**.0592**	.1561	.2810	.4571	.7581					

Heavy-face type indicates that the characteristic is 1 greater than before.

5. Convenient Equivalents

Unit	Equivalent	Logarithm	Unit	Equivalent	Logarithm
1 in.	2.540 cm.	0.404834	1 cu. ft.	7.48 gal.	0.873902
1 cm.	0.3937 in.	$\bar{1}$.595165	1 qt. (liquid)	57.75 cu. in.	1.761552
1 ft.	0.305 m. 30.48 cm.	$\bar{1}$.484300 1.484015	1 gal. (U. S.)	231 cu. in. 0.1337 cu. ft.	2.363612 $\bar{1}$.126131
1 m.	3.281 ft. 1.094 yd.	0.516006 0.039017	1 bbl. (U. S.)	31.5 gal. 4.21 cu. ft.	1.498311 0.624282
1 rd.	5.5 yd. 16.5 ft.	0.740363 1.217484	1 qt. (dry)	67.20 cu. in.	1.827369
			1 bu. (struck)	2150.4 cu. in. 1.244 cu. ft.	3.332519 0.094820
1 mi.	5280 ft. 1760 yd. 320 rd. 1609 m.	3.722634 3.245513 2.505150 3.206556	1 A.	160 sq. rd. 43,560 sq. ft.	2.204120 4.639088
1 km.	0.621 mi.	$\bar{1}$.793092	1 sq. mi. 1 section	640 A.	2.806180

6. Constants Relating to the Circle

Constant	Value	Logarithm	Constant	Value	Logarithm
π	3.1416	0.497151	$\frac{1}{2}\pi$	1.5708	0.196121
2π	6.2832	0.798181	$\frac{1}{3}\pi$	1.0472	0.020030
π^2	9.8696	0.994299	$\frac{1}{4}\pi$	0.7854	$\bar{1}$.895091
$\sqrt{\pi}$	1.7725	0.248587	$\frac{1}{6}\pi$	0.5236	$\bar{1}$.719000
$\sqrt{2\pi}$	2.5066	0.399085	$\frac{4}{3}\pi$	4.1888	0.622089

Circumference in	Value	Logarithm	Arc of	Length when $r=1$	Logarithm
Degrees	360	2.556303	1°	0.0174533	$\bar{2}$.241877
Minutes	21,600	4.334454	1′	0.0002909	$\bar{4}$.463744
Seconds	1,296,000	6.112605	1″	0.0000048	$\bar{6}$.681241

7. Legal Weights (Minnesota) for Standard Grains

Grain	Quantity	Weight	Grain	Quantity	Weight
Barley	1 bu.	48 lb.	Oats	1 bu.	32 lb.
Corn (shelled)	1 bu.	56 lb.	Rye	1 bu.	56 lb.
Flax	1 bu.	56 lb.	Wheat	1 bu.	60 lb.

Small grain occupies $1\frac{1}{4}$ cu. ft./bu.; ear corn occupies $2\frac{1}{2}$ cu. ft./bu.

8. Weights of Construction Materials

Material	Quantity	Weight	Material	Quantity	Weight
Birch	1 cu. ft.	45 lb.	Oak, white . . .	1 cu. ft.	46 lb.
Cement, Portland	1 sack	94 lb.	Pine, Norway .	1 cu. ft.	34 lb.
Cement, Portland	1 bbl.	376 lb.	Pine, white . .	1 cu. ft.	27 lb.
Concrete . . .	1 cu. ft.	140 lb.	Pine, yellow . .	1 cu. ft.	41 lb.
Fir, Douglas . .	1 cu. ft.	35 lb.	Sand	1 cu. ft.	105 lb.
Hemlock . . .	1 cu. ft.	30 lb.	Spruce	1 cu. ft.	28 lb.
Iron, cast . . .	1 cu. ft.	450 lb.	Steel	1 cu. ft.	490 lb.
Iron, wrought .	1 cu. ft.	480 lb.	Stone, crushed .	1 cu. ft.	100 lb.
Oak, red	1 cu. ft.	45 lb.	Tamarack . . .	1 cu. ft.	38 lb.

In measuring lumber, 12 ft. B. M. (board measure) = 1 cu. ft.

The equivalent volume of 1 sack of cement is 0.95 cu. ft.; that of 1 bbl. of cement is 3.8 cu. ft.

In estimating for coal bins, 35 cu. ft./T. is generally allowed for hard coal and 60 cu. ft./T. for coke.

9. Grades of Milk and Cream

1 Gal. at 68° F. of	Per Cent of Butter Fat	Per Cent of Non-Fat Solids	Per Cent of Total Solids	Specific Gravity	Weight in Pounds
Water . . .				1.000	8.3216
Milk, skim .	0.02	8.91	8.93	1.037	8.6295
Milk . . .	3.00	8.64	11.64	1.034	8.6045
Milk . . .	4.00	8.55	12.55	1.032	8.5879
Milk . . .	5.00	8.46	13.46	1.031	8.5796
Milk . . .	6.00	8.38	14.38	1.030	8.5713
Mixed . . .	10.00	8.02	18.02	1.025	8.5297
Cream . . .	15.00	7.57	22.57	1.018	8.4714
Cream . . .	18.00	7.31	25.31	1.015	8.4564
Cream . . .	20.00	7.13	27.13	1.013	8.4298
Cream . . .	25.00	6.68	31.68	1.008	8.3882
Cream . . .	30.00	6.24	36.24	1.004	8.3549
Cream . . .	35.00	5.79	40.79	0.999	8.3133
Cream . . .	40.00	5.35	45.35	.995	8.2800
Cream . . .	45.00	4.90	49.90	.990	8.2467
Cream . . .	50.00	4.45	54.45	.986	8.2067

The above table (after Washburn) shows the specific gravity, the weight per gallon at 68° F., and the distribution of solids in milk and cream graded according to the per cent of butter fat.

10. Friction Factors (f) for Clean Iron Pipes

Diameter (d) of Pipe in Inches	Velocity (v) in Feet per Second						
	1	2	3	4	6	10	15
$\frac{1}{2}$	0.049	0.043	0.038	0.035			
$\frac{5}{8}$	.047	.041	.037	.034	0.031	0.029	0.028
$\frac{3}{4}$	.045	.039	.036	.033	.030	.028	.027
1	.042	.036	.033	.030	.027	.026	.024
$1\frac{1}{4}$	.038	.032	.030	.028	.026	.024	.023
$1\frac{1}{2}$	.037	.031	.029	.028	.026	.024	.023
2	.035	.030	.028	.027	.025	.023	.022
$2\frac{1}{2}$	.034	.029	.027	.026	.024	.022	.021
3	.032	.028	.026	.025	.024	.022	.021

The above table is adapted from Table 33, in *Treatise on Hydraulics* by Merriman. For values of v not given in the table, interpolate by proportion.

11. Convenient Equivalents in Hydraulic Problems

Unit	Equivalent	Logarithm
1 cu. ft. of water at 0° C.	62.5 lb.	1.795880
1 gal. of water at 0° C.	8.355 lb.	0.921946
1 ft. of hydrostatic head	0.434 lb./sq. in. of pressure	$\bar{1}$.637490
1 lb./sq. in. of pressure	2.304 ft. of hydrostatic head	0.362482
1 acre-inch of water	3630 cu. ft. 27,154 gal.	3.559907 4.433834
Runoff of $\frac{1}{4}$ in. from 1 A. in 24 hr.	0.0105 cu. ft./sec.	$\bar{2}$.021189
Acceleration of gravity (g)	32.2	1.507856
$\sqrt{2g}$	8.02	0.904174

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